Geometric interpretations for quantum invariants

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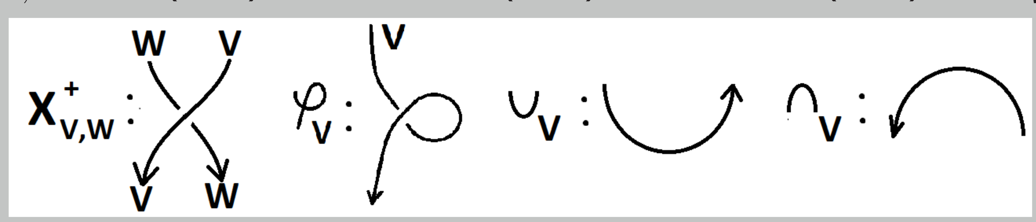
Quantum invariants

In 1991, Reshetikhin and Turaev defined a construction that, starting with any ribbon category $\mathcal{C}$, produces invariants for $\mathcal{C}$-coloured links. A ribbon category is a monoidal category with a collection of morphisms between any pair of objects $C_{V,W}$ and also isomorphisms and dualities which are compatible.

Theorem (Reshetikhin-Turaev):
Let $(\mathcal{C}, C, \Theta, b, d')$ be a ribbon category. Then there exists a unique functor defined on the category of coloured tangles $F_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$ which is monoidal and satisfies the following local relations for any $V, W \in \mathcal{C}$:

(1) $F((V,+)) = V \qquad F((V,-)) = (V)^*$

(2) $F(X^+_{V,W}) = C_{V,W} \quad F(\varphi_V) = \Theta_V \quad F(\cup_V) = b_V \quad F(\cap_V) = d'_V$ where



$\mathcal{U}_q(\mathfrak{sl}_2)$ and the Jones polynomial

For $\mathfrak{sl}(2)$, the representation theory of its quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ will lead to two kinds of invariants: the coloured Jones polynomial and the ADO invariant, depending on whether q is root of unity or not. Let $q \in \mathbb{C}$ be not a root of unity. Then the simple finite dimensional representations V_N of $\mathcal{U}_q(\mathfrak{sl}_2)$ are indexed by their dimension $N \in \mathbb{N}$. This algebra has an extra structure, an R-matrix with a well-defined action on $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$, which leads with the previous construction to link invariants, the coloured Jones polynomials:

$$J_N(L)(q) = RT(V_N, \ldots, V_N; L)$$

For $N = 2$ this is the classical Jones polynomial: $J_2(L)(q) = (-1)^{\sharp L - 1} J(q^{-1})$

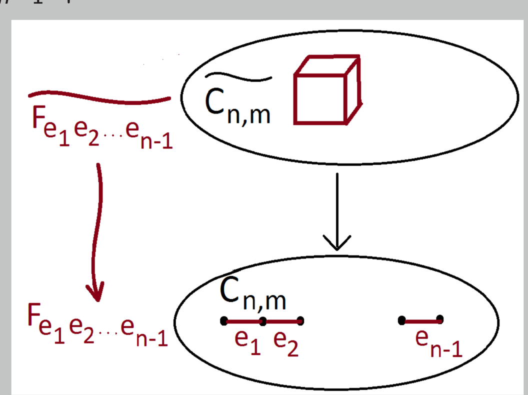
Lawrence representation

In [Law90] Lawrence introduced a homological representation of the braid group B_n. Let $D_n = \mathbb{D}^2 - \{1, 2, \ldots, n\}$ and $C_{n,m} = \mathrm{Conf}_m(D_n)$. Then $H_1(C_{n,m}) \cong \mathbb{Z}^n \oplus \mathbb{Z}$. Consider the local system

$$\varphi \colon \pi_1(C_{n,m}) \longrightarrow H_1(C_{n,m}) \xrightarrow{\text{augmentation}} \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle d \rangle$$

and $\tilde{C}_{n,m}$ the corresponding covering. Let

$$H_{n,m} = \langle F_{e_1,\ldots,e_{n-1}} \mid e_1 + \cdots + e_{n-1} = m \rangle \;\subseteq\; H^{\mathrm{lf}}_m(\tilde{C}_{n,m})$$



Since $B_n = \mathrm{MCG}(D_n)$, it will induce a well-defined action on $H^{\mathrm{lf}}_m(\tilde{C}_{n,m})$, and $\varphi_{n,m} \colon B_n \to \mathrm{End}(H_{n,m})$ is called the *Lawrence representation*.

Bigelow-Lawrence representation

In [Big02] Bigelow, following Lawrence, gave a homological description for the Jones polynomial. The space is $C_{2n,n}$ and the local system is constructed in a similar way, $\phi \colon \pi_1(C_{2n,n}) \to \mathbb{Z}\langle q \rangle \oplus \mathbb{Z}\langle t \rangle$. There is a duality on the covering, using deck transformations, called the *Blanchfield form*:

$$\langle -, - \rangle \colon H^{\mathrm{lf}}_n(\tilde{C}_{2n,n}) \otimes H_n(\tilde{C}_{2n,n}, \partial) \longrightarrow \mathbb{Z}[q^{\pm 1}, t^{\pm 1}].$$

There exist certain submanifolds $S, T \subseteq C_{2n,n}$ that may be lifted to $\tilde{S}, \tilde{T} \subseteq \tilde{C}_{2n,n}$ and that determine homology classes. Again considering the braid group as a mapping class group we have:

Theorem (Bigelow, Lawrence): For a link L with $L = \hat{\beta}$ for $\beta \in B_{2n}$,

$$J(L)(q) = \mathrm{ct}_q(\mathrm{width}(\beta)) \cdot \langle \tilde{S}, \beta \tilde{T} \rangle |_{t = -q^{-1}}$$

References

- [ADO92] Y. Akutsu, T. Deguchi and T. Ohtsuki, *Invariants of colored links*, Journal of Knot Theory and its Ramifications 1 (1992) 161–184.
- [Big02] Stephen Bigelow, *A homological definition of the Jones polynomial*, Geometry & Topology Monographs 4 (2002) 29–41.

$\mathcal{U}_q(\mathfrak{sl}_2)$ at roots of unity and the ADO polynomial

In [ADO92], Y. Akutsu, T. Deguchi and T. Ohtsuki defined a renormalized-type invariant from $\mathcal{U}_q(\mathfrak{sl}_2)$ at roots of unity. Fix $N \in \mathbb{N}$ and let $q = e^{\frac{2\pi i}{2N}}$. The N-dimensional representations V_c of $\mathcal{U}_q(\mathfrak{sl}_2)$ are indexed by the complex numbers $c \in \mathbb{C}$. This category of representations has a braiding and dualities that allow one to apply the Reshetikhin-Turaev construction for links (looking at a link as a morphism from $\varnothing$ to $\varnothing$).

The subtle problem with directly applying the RT construction in this way is that the resulting invariant is zero, due to the algebraic properties of the algebra at roots of unity, namely from the vanishing of the quantum dimension. The method of [ADO92] solves this by cutting the link at one strand, using the functorial RT construction for the resulting $(1,1)$-tangle and then multiplying with a correction factor in order to obtain an invariant.

Remark: Due to the form of the R-matrix of $\mathcal{U}_q(\mathfrak{sl}_2)$, instead of working over the ring $\mathbb{C}$, one may instead work over the ring $\mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}]$, where q is a root of unity and $\lambda \in \mathbb{C}$.

Theorem ([ADO92]): Let T be a $(1,1)$-tangle coloured by $V_{\lambda_1}, \ldots, V_{\lambda_n}$ (for $\lambda_i \in \mathbb{C}$) with the strand corresponding to V_{λ_1} open, and let $L = \hat{T}$ be the corresponding coloured link. Then

$$\Phi_N(L; \lambda_1, \ldots, \lambda_n) := \{\lambda_1 + N, N - 1\}^{-1} RT(V_{\lambda_1}, \ldots, V_{\lambda_n}; T)$$

is a well-defined (non-trivial) link invariant.

Truncated Lawrence representations

We remain in the case where q is a root of unity. Fix $d = -q^2$ and define

$$H^{\geqslant N}_{n,m} := \langle F_{e_1,\ldots,e_{n-1}} \mid e_1, \ldots, e_{n-1} \in \mathbb{N}, e_1 + \cdots + e_{n-1} = m, \ \exists i : e_i \geqslant N \rangle \subseteq H^{\mathrm{lf}}_m(\tilde{C}_{n,m})$$

$$\bar{H}_{n,m} := H_{n,m} / H^{\geqslant N}_{n,m}$$

Proposition (Ito [Ito15]):
The action of $B_n = MCG(D_n)$ preserves $H^{\geqslant N}_{n,m}$ and induces a representation $\bar{\varphi}_{n,m} \colon B_n \to \mathrm{End}(\bar{H}_{n,m})$, called the *truncated Lawrence representation*.

Relations between algebraic and homological representations

Theorem (Khono, Ito):

(1) Let q be generic and let λ be *fully generic w.r.t. q*, meaning that $1, q, q^{\lambda}$ are algebraically independent. Let V_{λ} be the canonical infinite-dimensional representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ of weight q^{λ}. Then the RT-representation $RT_{V_{\lambda},\ldots,V_{\lambda}} \colon B_n \to \mathrm{End}(V_{\lambda}^{\otimes n})$ splits as $\bigoplus W_{n,m}$ as a $\mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}][B_n]$-representation. The Lawrence representation $\varphi_{n,m}$ is a $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}][B_n]$-representation, which we may also consider as a $\mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}][B_n]$-representation by induction along the ring homomorphism $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}] \to \mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}]$ sending x to $q^{-2\lambda}$ and d to $-q^2$. As such, $W_{n,m}$ and $\varphi_{n,m}$ are isomorphic. Summarising, we may write: $\beta \curvearrowright W_{n,m} \simeq \varphi_{n,m}(\beta) \mid x = q^{-2\lambda}, d = -q^2$, for all $\beta \in B_n$.

(2) For q a $2N$-th root of unity, similarly: $RT_{V_{\lambda},\ldots,V_{\lambda}}$ splits as $\bigoplus U_{n,m}$ as a $\mathbb{Z}[q^{\pm 1}, q^{\pm \lambda}][B_n]$-representation and we have: $\beta \curvearrowright U_{n,m} \simeq \bar{\varphi}_{n,m}(\beta) \mid x = q^{-2\lambda}, d = -q^2$, for all $\beta \in B_n$.

Geometrical interpretations for ADO: Further directions

As we have seen, both ADO and Jones invariants are quantum invariants arising from $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$. For the Jones polynomial, that had a purely algebraic description, using the Lawrence homological representation: it was described as a pairing of submanifolds in a certain space. ADO has a different flavour, being a renormalised invariant, but the relation between the algebraic Reshetikhin-Turaev construction that defines it and the truncated Lawrence representation suggests that it might be possible to describe it in a similar way.

References

- [Ito15] Tetsuya Ito, *A homological representation formula of colored Alexander invariants*, arxiv.org/abs/1505.02841 (2015).
- [Law90] Ruth J. Lawrence, *Homological representations of the Hecke algebra*, Commun. Math. Phys. 135 (1990) 141–191.

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