UNIVERSITY OF BUCHAREST
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

BACHELOR THESIS

ALEXANDER POLYNOMIALS OF THREE MANIFOLDS

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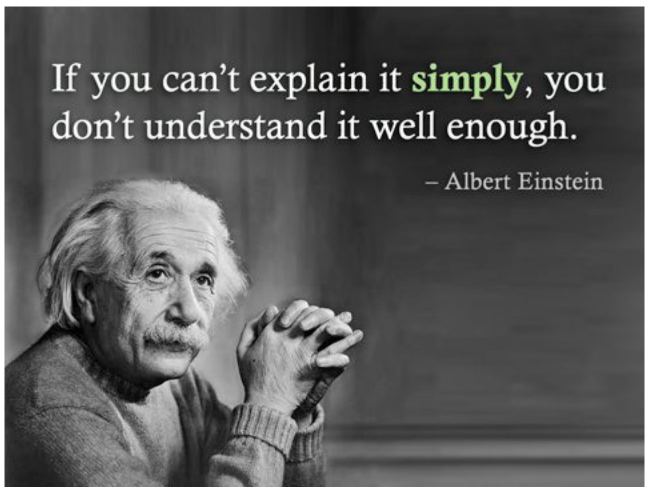
I have reserved the last but not least word from these acknowledgements, for the person who has determined my math trajectory, how it looks like at this moment: Prof. Liviu Ornea. Along these years, his lectures were for me, a great scientific experience. His Algebraic Topology course, I think, was a key moment for the decision to follow a topological direction for my studies.

...in simple words...



Rezumat

Subiectul acestei lucrări îl constituie teoria nodurilor. Acest domeniu a apărut odata cu lucrările lui Gauss legate de numere de înlănţuire. Ca ramură distinctă a topologiei s-a constituit la începutul secolului XX prin lucrările lui Poincare, Alexander si Dehn.

În anul 1928 este introdus pentru prima data aşa numitul polinom Alexander. Acest invariant, este suficient de puternic pentru a detecta diferenţe inaccesibile fără el, dar totodată relativ limitat: de exemplu nu poate detecta diferenţa dintre două noduri care sunt unul imaginea în oglinda a celuilalt.

Acest neajuns este rezolvat parţial în anii '90 odată cu apariţia polinomului Jones şi a întregii pleiade de invarianţi care au apărut ulterior (invarianţi cuantici, invarianţi polinomiali in 2 variabile, etc).

Scopul acestei lucrări este de a prezenta pe de o parte polinomul Alexander sub multiplele sale faţete iar pe de alta de a studia relaţia dintre varietaţile caracteristice şi reprezentările metabeliene ale grupului unui nod.

Lucrarea este constituită din 4 capitole, o introducere şi un apendix destinat descrierii unor noţiuni şi teoreme necesare pe parcursul tezei.

In cadrul introducerii este prezentat subiectul general al teoriei nodurilor şi metodele de abordare ale problemelor prin prisma teoriei Alexander.

Primul capitol este destinat introducerii principalelor ingrediente utilizate de-a lungul tezei: modulul si polinomul Alexander. Totodată sunt introduse şi varietăţile caracteristice care vor juca un rol esenţial in cadrul ultimei părţi a lucrării.

În cadrul capitolului 2 sunt prezentate o serie de rezultate cu privire la topologia complementelor de noduri. Astfel, primele două paragrafe se referă la proprietăţile omologice şi omotopice ale complementelor. În continuare sunt prezentate diagramele planare, mişcările Reidemeister, relaţia dintre linkuri şi braiduri, precum şi mişcările Markov. O atenţie deosebită este acordată în paragraful 5, metodelor de construcţie a braidurilor asociate

linkurilor: se demonstrează teorema Alexander şi se descrie algoritmul Vogel. Ultimele paragrafe ale acestui capitol se referă la grupul fundamental al complementului în cele două abordări uzuale: Wirtinger şi cea care utilizează grupul braid, mai precis scufundarea acestuia în grupul de automorfisme ale grupului liber.

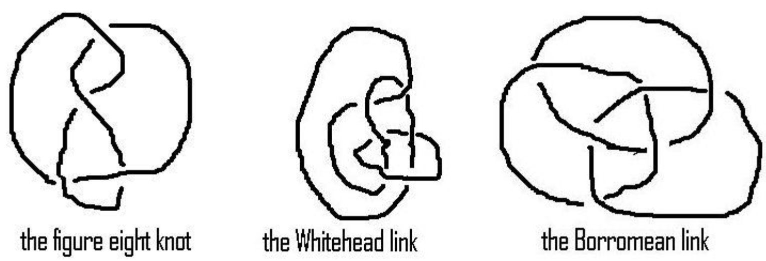
Următorul capitol se referă în exclusivitate la trei dintre modalităţile de construcţie ale polinomului Alexander: prin calcul Fox, via grupuri braid şi cu relaţii skein. Este descrisă reprezentarea Burau şi demonstraţia existenţei polinomului Alexander-Conway. Din păcate am omis descrierea contrucţiei care foloseşte suprafaţa Seifert. În ultimul paragraf sunt prezentate proprietăţile generale ale polinomului Alexander, inclusiv relaţiile Torres.

Capitolul 4 este destinat aplicaţiilor. În prima parte (paragrafele 1 si 2), sunt prezentate colorările Fox, relaţia dintre colorări si proprietăţi aritmetice ale polinomului Alexander, precum şi o clasă specială de linkuri: cele cu 2 poduri (2-bridge links). Cu privire la acestea din urmă, este prezentat grupul fundamental şi proprietăţile speciale ale polinomului Alexander. În paragraful 3 se prezintă relaţia dintre varietăţile caracteristice şi anumite clase de reprezentări metabeliene. Rezultatele principale ale acestui paragraf sunt teoremele 4.3.8 şi 4.3.9 din [19] în care se calculează cardinalul morfismelor, epimorfismelor precum şi invarianţii Hall în cazul unor extinderi metaciclice. Finalul acestui paragraf este destinat unui rezultat similar, pentru cazul morfismelor de la grupul unui nod cu 2 poduri într-un grup diedral.

Lucrarea se încheie cu o scurtă descriere a trei potenţiale direcţii de continuare a studiului început în această teză.

Introduction

The main objects we are talking about in this thesis are (oriented) knots K and links L in $\mathbb{R}^3$ or $\mathbb{S}^3$. Below are three examples: the figure eight knot, the Whitehead link and the Borromean link (all pictures use the [28] on-line converter).



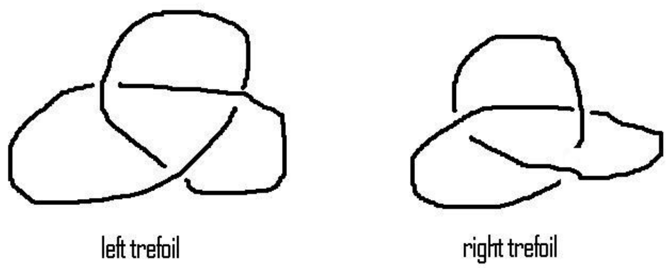
We will deal here only with polygonal or smooth links and it is known the following:

Theorem *Every smooth link is isotopic with a polygonal one.*

Two links are considered the same (equivalent), denoted by $L \simeq L'$ if they are ambient isotopic. An equivalent assumption is that there exists a homeomorphism of $\mathbb{S}^3$, which preserves the orientation and takes L to L' cf. [16] . For smooth links, ambient isotopy is the same as isotopy cf. **A2**. A coarser equivalence relation is:

Definition *Two links are weakly-equivalent, denoted by $L \sim L'$ if there is a homeomorphism of $\mathbb{S}^3$ taking one in the other.*

The difference is fundamental. For example the two trefoils

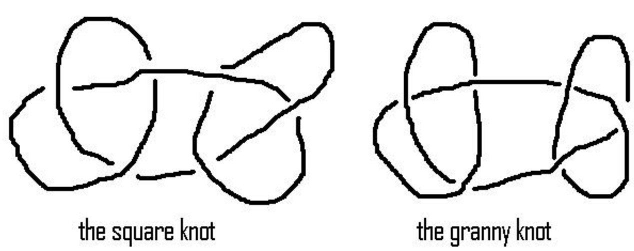


are weakly equivalent but not equivalent. The aim of knots/links theory is to classify them up to (weak)-equivalence. A surprisingly quite recent result is the following theorem due to Gordon and Luecke. It appeared first in [8] and with full details in [9]:

Theorem *Two knots with homeomorphic complements are weakly equivalent.*

Remark *First of all the two trefoils have homeo complements but they are not equivalent. For example the Jones polynomial distinguishes them. Secondly, the above theorem is not true for links !*

However, even if the weak-equivalence for knots is considered, the homeo type of the complement is a hardly tractable invariant. It is very desirable to have weaker invariants which are easy to compute, possibly by combinatorial techniques. One such invariant is the fundamental group of the complement: $\pi = \pi_1(\mathbb{S}^3 \backslash K)$. Although π distinguishes the unknot, it is not a complete invariant for weak-equivalence. For example, the square and granny knots



have the same π but are not weak-equivalent. In fact the peripheral structure distinguishes them.

π is non-abelian except when the knot is trivial and usually given via a presentation by generators and relations. It is a hard problem to decide when two presentations give isomorphic groups.

The next step toward a computable invariant is the so-called Alexander module M of the knot group. For a presentation of the group, there are several methods to produce presentations for this module. Then, using Fitting ideals, the (computable) Alexander polynomial invariant is obtained.

Chapter 1

Alexander polynomials of groups and spaces

The aim of this chapter is to associate in topological and algebraic contexts tractable invariants: modules, varieties, polynomials. The main references are [18], [4], [21] and [25].

1.1 Alexander modules

Let X a connected CW-complex of finite type, with only one 0-cell x_0 and $\pi = \pi_1(X, x_0)$. Let $H = \pi_{ab} = H_1(X, \mathbb{Z})$. In some situations we can use complex coefficients instead of integers. From the standard correspondence between coverings of X and subgroups in π, associated to the surjection $\pi \to H$ there is a covering $p : \tilde{X} \to X$ named the maximal abelian covering. In such a situation the homology of the covering became a module over the group of deck transformation, wich, in this setting, is nothing else than H. This H-module structure on $H_1(\tilde{X}, \mathbb{Z})$ became a true module structure over $\mathbb{Z}H$ i. e. the group ring of H. We summarize with the following:

Definition 1.1.1 *The $\mathbb{Z}H$- module $B_\pi := H_1(\tilde{X}, \mathbb{Z})$ is named the Alexander invariant of X.*

Definition 1.1.2 *The $\mathbb{Z}H$- module $A_\pi := H_1(\tilde{X}, p^{-1}(x_0)\mathbb{Z})$ is named the Alexander module of X.*

The previous structure can be considered in a purely algebraic setting. Let G be a group, G' its commutator and G_{ab} its abelianisation. Let G'' the commutator of G'. Using the exact sequences:

$$0 \to G' \to G \to G_{ab} \to 0$$

$$0 \to \tfrac{G'}{G''} \to \tfrac{G}{G''} \to G_{ab} \to 0$$

it can be proved the following:

Lemma 1.1.3 *The pseudo-action by conjugation of G_{ab} on G' induce a well-defined action on $\frac{G'}{G''}$.*

In the lemma above the term pseudo-action is used because the conjugation is not well-defined on G'. It becomes well-defined only after factorization. As in the topological case, by passing to the group ring we obtain a $\mathbb{Z}G_{ab}$-module structure on $\frac{G'}{G''}$. In this algebraic setting it is called the Alexander invariant of G and is denoted by B_G [4]. In the case where $G = \pi_1(X)$ both constructions give the same answer, i.e. with the notations above and from the previous section, we have the following identification between the topological and algebraic Alexander invariant of X respective π [25]:

Theorem 1.1.4 *There is a natural isomorphism of $\mathbb{Z}G_{ab}$- modules*

$$\tfrac{G'}{G''} \simeq H_1(\tilde{X}, \mathbb{Z}).$$

Also, the homology sequence for the pair $(\tilde{X}, p^{-1}(x_0))$ gives

$$0 \to H_1(\tilde{X}) \to H_1(\tilde{X}, p^{-1}(x_0)) \to H_0(p^{-1}(x_0)) \to \mathbb{Z},$$

the last map being the augmentation $\epsilon : H_0(p^{-1}(x_0)) \to H_0(\tilde{X}) \simeq \mathbb{Z}$. In Alexander-type terms, we obtain the following exact sequence,

$$0 \to B_G \to A_G \to I \to 0$$

where I is the kernel of the evaluation map $\epsilon : \mathbb{Z}G_{ab} \to \mathbb{Z}$. If we identify $\mathbb{Z}G_{ab}$ with $\Lambda = \mathbb{Z}[{t_1}^{\pm 1}, ..., {t_q}^{\pm 1}]$, then $I = (t_1 - 1, ...t_q - 1)$ and

$$A_G = \mathbb{Z}G_{ab} \otimes I.$$

1.2 Alexander varieties

For X, G as above, we have the following:

Definition 1.2.1 *For $k \geq 1$ the k^{th} Alexander (characteristic) variety of X (or G) is the $(k+1)^{th}$ support variety of the Alexander module:*

$$\mathcal{V}_k(G) = V_{k+1}(A_G \otimes \mathbb{C}) \cup \{\mathbb{I}\},$$

where $\mathbb{I}$ is the identity (trivial representation) in the character torus $\mathbb{T}$. From the definition above, it is clear that the characteristic varieties form a decreasing sequence (as the $E's$ form an ascending one cf. appendix **A3**) and that they depend only on $\frac{G}{G''}$. Also we have the following important remark which relates the support varieties of various levels of the Alexander module with to those of the Alexander invariant:

Remark $V_{k+1}(A_G \otimes \mathbb{C}) = V(E_k(A_G \otimes \mathbb{C})) = V(E_{k-1}(B_G \otimes \mathbb{C})) = V_k(B_G \otimes \mathbb{C})$.

1.3 Alexander polynomials

For X and G as above, with $q = b_1(X) \geq 1$ we have:

Definition 1.3.1 *The Alexander polynomial of X (or G) is*

$$\Delta_X = \Delta_G = \Delta_1(A_G).$$

Remark *It depends only on G (in fact only on $\frac{G}{G''}$) and is defined up to multiplication by units in Λ.*

For a group G, the deficiency, $def(G)$, is the minimum of the difference between the number of generators and relations, over all presentation. The following theorem from [5] expresses the first elementary ideal of the Alexander module, in terms of the Alexander polynomial when the deficiency is positive:

Theorem (Eisenbud-Neumann) 1.3.2 *If $b_1(X) = 1$ then*
$E_1(\Delta_G) = (\Delta_G)$ (i.e. it is a principal ideal).
If $b_1(X) \geq 2$ and $def(G) > 0$ then $E_1(A_G) = I \cdot (\Delta_G)$.

The next result shows the relation between the Alexander polynomial and the characteristic varieties:

Proposition 1.3.3 *$\Delta_G = 0$ iff $\mathcal{V}_1(G) = \mathbb{T}_G$. If $\Delta_G \neq 0$ then*

$$\mathcal{W}_1(G) = \begin{cases} V(\Delta_G) & \text{if } q > 1 \\ V(\Delta_G) \cup \{\mathbb{I}\} & \text{if } q = 1 \end{cases}$$

If $q \geq 2$ then $\mathcal{W}_1(G) = \emptyset$ iff Δ_G is constant.

Corollary 1.3.4 *If $def(G) > 0$ then $\mathcal{V}_1(G) = V(\Delta_G) \cup \{\mathbb{I}\}$.*

Chapter 2

Knots and links in 3-manifolds

In the following K and L denote a knot or a link in $\mathbb{S}^3$. X is the complement. For L we shall denote by $K_1, ..., K_m$ the knot components of L. N is a tubular neighborhood of L; it is the union of m closed solid tori. $E = \mathbb{S}^3 \backslash Int(N)$. Obviously E is a 3-manifold with boundary, X is an open 3-manifold, they have the same homotopy type and consequently the same π_1 and the same homology. In fact E is a deformation retract of X.

2.1 Knots and links complements: (co) homological properties

The next two theorems describe the (co)homological structure of the link complement. They show that the homological information is totally insensitive to knotting. Among the main references there are [22] and [6].

Theorem 2.1.1 *$H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^m$, $H_2(X) = \mathbb{Z}^{m-1}$ and $H_i(X) = 0$ for $i \geq 3$.*

On the cohomological side we have:

Theorem 2.1.2 *$H^0(X) = \mathbb{Z}$, $H^1(X) = \mathbb{Z}^m$, $H^2(X) = \mathbb{Z}^{m-1}$ and $H^i(X) = 0$ for $i \geq 3$.*

2.2 Knots and links complements: homotopical properties

Definition 2.2.1 *For a group π and a positive integer n by $K(\pi, n)$ we denote the unique homotopy type of spaces (called Eilenberg-Maclane) for which the only nonzero homotopy group is π in dimension n.*

From the homotopical point of view, the first main result is [2]:

Theorem 2.2.2 *For a knot, the complement is an Eilenberg-Maclane $K(\pi, 1)$ space.*

For links, with the following:

Definition 2.2.3 *A link L is split if it can be decomposed as $L_1 \cup L_2$ with the L_i's in the interior of disjoint 3-balls.*

we have:

Theorem 2.2.4 *For a link, the complement is an Eilenberg-Maclane $K(\pi, 1)$ space iff L is not split.*
If $L = L_1 \cup ... \cup L_k$ with the L_i's nonsplit, then the complement X is homotopic equivalent with $K(\pi, 1) \vee \mathbb{S}^2 \vee ... \vee \mathbb{S}^2$, where the number of spheres is $k-1$.

We now turn to the peripheral system and semi-direct product structure of knot/link groups. First of all we want to mention the following theorems:

Theorem 2.2.5 *A knot is trivial iff π_1 is infinite cyclic.*

Theorem 2.2.6 *A knot is nontrivial iff $\pi_1(\partial V) \to \pi_1(X)$ is injective.*

From the previous theorem every nontrivial (different from $\mathbb{Z}$) knot group has a specified subgroup isomorphic to $\mathbb{Z}^2$. This subgroup is named the peripheral structure on π. It is defined only up to conjugation. A celebrated theorem of Waldhausen [2] is the following:

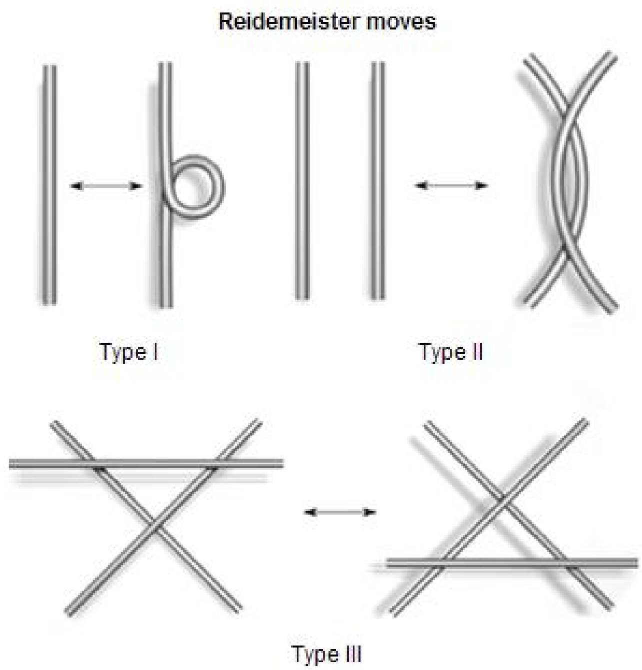
Theorem 2.2.7 *A knot is determined up to equivalence by the isomorphism class of its peripheral structure.*

For example [25], Fox used this theorem to prove that the square and granny knots, even having the same π, are distinct.

Another important fact about knot groups is that the surjection $\pi \to \pi_{ab}$ has a section which sends the generator (i. e. the homology class of a meridian) into its homotopy class. It follows that the knot group is in fact a semidirect product $\pi' \rtimes \pi_{ab}$.

2.3 Links, planar diagrams and Reidemeister moves

In this section we will explain the representation of links by planar diagrams. Choosing a generic plane in the 3-space and projecting the link on it we obtain a figure consisting of a number of crossings. We should always consider projections with only transverse crossings. Of course, the number of crossings or the succession of their type along a component of the link are by no means invariants of the link. One major problem concerning the projections is to decide when two of them represent the same link. The following type of moves called Reidemeister moves do not change the isotopy type of the link:



The main point is that we have the following remarkable theorem [23]:

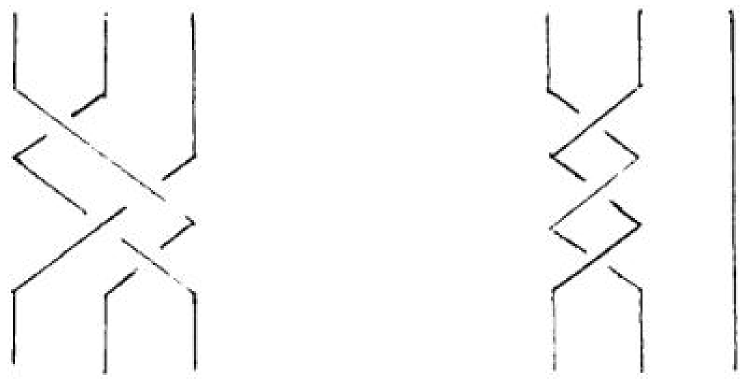
Theorem 2.3.1 *Two planar link diagrams represents equivalent links iff they can be transformed one in the other by Reidemeister moves.*

2.4 Links, braids and Markov moves

This section is devoted to braids and their relations with links cf. [1] [22].

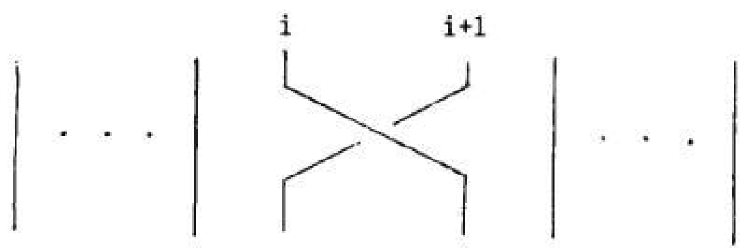
Definition 2.4.1 *A n-braid consist of:*
1. n points in $\mathbb{R}^3$ with the z-coordinate a and the x coordinate strictly increasing denoted by P_i
2. n points in $\mathbb{R}^3$ with the z-coordinate b and the x coordinate strictly increasing denoted by Q_i
3. A permutation ϵ and for every i a path from P_i to $Q_{\epsilon(i)}$, such that on every path the z-coordinate is strictly decreasing.
3. $a < b$ and the paths are disjoints.

In the picture below, taken from [22] there are 2 examples of 3-braids:



As for links, there is a notion of equivalent braids up to isotopy; the conditions $1-4$ must be verified at each moment and the the vertices are fixed through the isotopy. Even if at first sight some conditions from above might be redundant, it is not the case. For example, the following two braids from [22] are not isotopic:

The operation of gluing two n-braids define a group structure on the classes of equivalent braids: the Artin braid group B_n. A celebrated theorem of Artin asserts that if we denote by σ_i the following braid (picture from [22])



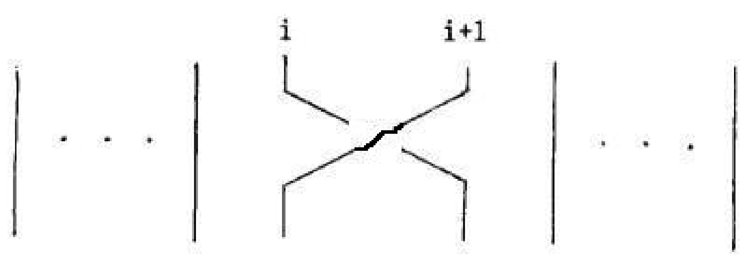
then:

Theorem 2.4.2 *The σ_i's for $i = 1...n-1$ are a generator system for B_n with the relations:*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \textit{ for } \mid i-j \mid \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \textit{ for } 1 \leq i \leq n-2.$$

For example, in the group B_n the inverse of σ_i is simply



Another important result due also to Artin is the representation of B_n in the automorphisms group of F_n, the free groups on n letters $x_1, ..., x_n$. Let ξ_i the autmorphism of F_n defined as follows:

$$\xi_i(x_i) = x_i x_{i+1} {x_i}^{-1}$$

$$\xi_i(x_{i+1}) = x_i$$

$$\xi_i(x_j) = x_j \text{ for } j \neq i, \ i+1.$$

Artin representation theorem asserts that:

Theorem 2.4.3 *1. The map $\varphi : B_n \to Aut(F_n)$ defined on generators by: $\varphi(\sigma_i) = \xi_i$ is a well defined injective morphism from B_n to $Aut(F_n)$.*
2. An element $\xi \in Aut(F_n)$ is in the image of φ iff there are words $A_i \in F_n$ and ϵ a permutation such that:

$$\xi(x_i) = A_i x_{\varepsilon(i)} {A_i}^{-1} \text{ for } 1 \leq i \leq n$$

$$\xi(x_1 ... x_n) = x_1 ... x_n.$$

Proof: The main point is to define for each $\sigma \in B_n$ an explicit automorphism $\bar{\sigma}$ of F_n. We consider a point P in the $z = a$ plane with the x-coordinate smaller than any x-coordinate of the P_i's and also that its projection Q on the $z = b$ plane has the same property in relation with the Q_i's. Consider the ambient space $\mathbb{R}^3$ with the strings of σ removed. The plane $z = a$ became a plane with n-points removed, having hence the fundamental group F_n. We can think the generators x_i's as loops around the P_i's based at P. We push down each x_i to a loop in the $z = b$ plane based at Q. Because the $z = b$ plane with the Q_i's removed has the same fundamental group F_n, this push down operation is in fact a map $\bar{\sigma} : F_n \to F_n$. The theorem is proved along the followings small steps:
1. if l_1 and l_2 are P based loops in the pointed $z = a$ plane, then $\bar{\sigma} l_1$ and $\bar{\sigma} l_2$ are homotopic loop in the pointed $z = b$ plane;
2. if l_1 and l_2 are P based loops in the pointed $z = a$ plane, then $\bar{\sigma}(l_1 l_2) = \bar{\sigma} l_1 \bar{\sigma} l_2$;
3. $\bar{\sigma}$ is bijective; in fact it has as inverse the push up operation;
From the above steps $\bar{\sigma}$ is an automorphism.
4. if τ is another braid isotopic with σ then $\bar{\sigma} = \bar{\tau}$ (in fact, according to the next step it is sufficient to prove this for a braid isotopic with the trivial one);
5. $\bar{\sigma\tau} = \bar{\sigma}\bar{\tau}$;
6. $\bar{\sigma}(x_1 ... x_n) = x_1 ... x_n$;
7. for $\sigma = \sigma_i$, $\bar{\sigma}_i$ verify $\bar{\sigma}_i(x_i) = x_i x_{i+1} {x_i}^{-1}$, $\bar{\sigma}_i(x_{i+1}) = x_i$
and $\bar{\sigma}_i(x_j) = x_j$ for $j \neq i,\ i+1$;
8. as $B_n =< \sigma_1, ..., \sigma_n >$ and $\bar{\sigma}$ is an automorphism of F_n then $\bar{\sigma} x_i$ has the form $A_i x_{\varepsilon(i)} {A_i}^{-1}$ for A_i's words in F_n and ε a permutation in S_n (induction on the length of σ as word in the σ_i's and their inverses).
It remains to show that a morphism of the above form is induced by a braid.

The proof is by induction on $l = \Sigma l(A_i)$, the sum of the lengths of the A_i's in B_n.
For $l = 1$ there is nothing to prove: ξ is the identity. Suppose the assertion true for $l < m$ and let a ξ with $l = m$. By multiplication of all the relations $\xi(x_i) = A_i x_{\varepsilon(i)} {A_i}^{-1}$, the left hand side has length n and so some cancellations must occur on the right. A careful analysis of this process (cf. [22] pp. 90) together with the inductive hypothesis finishes the proof. **QED**

In the end of this section we discuss the relation between links and braids. First of all there is a natural operation of "closing" a braid, producing a link. A fundamental result due to Alexander asserts the converse: any link is the closure of a braid. We will talk about it in the next section.

An important question is when two braids give the same isotopy class of links. As in the case of planar representation of links, there are two type of moves on braids which leave the associated link invariant:
1. the first is simply conjugation by another braid
2. for the second, if the initial braid b is in B_n, we shall ebbed b in B_{n+1} by adding one string and the move is simply multiplication by $\sigma_n^{\pm 1}$. The following theorem due to Markov, gives the complete answer to the above question.

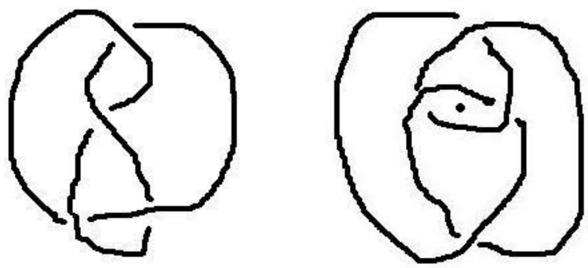
Theorem 2.4.4 *Two braids give equivalent links iff they are related by a finite number of Markov moves.*

2.5 Alexander trick and Vogel algorithm

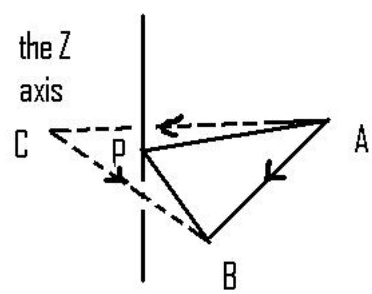
The main references for this section are Prasolov and Sossinsky [23], Kassel and Turaev [14] and Manturov [17]. It is devoted to Alexander theorem below; the first proof we present uses the so called Alexander trick. The second consists of the Vogel algorithm.

Theorem 2.5.1 *Any link is the closure of a braid.*

Proof: Consider a polygonal oriented link L in $\mathbb{R}^3$ (recall that any tame link is isotopic with a polygonal one) such that any edge is not perpendicular on the horizontal plane. An edge is named positive if its projection on the plane points counterclockwise viewed from the origin, and negative if not. By the chosen position of L, any edge is positive or negative. If all the edges are positive, we take the projection on the plane (such a projection is named braided) and then cut the plane along a half-line from the origin; we obtain the desired braid. For example, in the picture below, we have two projection of the figure eight knot: the second is braided, while the first is not:



Suppose we have a negative edge AB. If there is a point P on the z-axis such that the triangle PAB intersects L only along AB (such an edge is named accessible), then we can take a point C such that:
-the triangle ABC cuts L only along AB
-the triangle ABC contains P.
By replacing AB with AC and CB as in the picture below, we arrive at a diagram for the same link but with fewer negative edges.



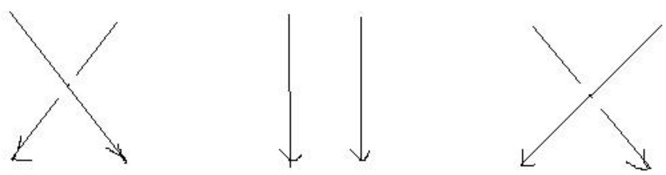
The negative edge AB replaced by AC and CB

If AB is not accessible, however any point in it is contained in an accessible sub-segment. By compactness of AB, it can be subdivided in a finite number of negative but accessible edges. These can be replaced by some positives with the previous method. We arrive again at a diagram with fewer negative edges. **QED**

We turn now to the Vogel algorithm. Recall from the proof above the following:

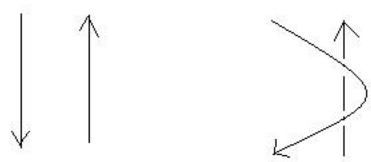
Definition 2.5.2 *A planar diagram D for an oriented link L is called braided if there exists a point O in the plane from which all edges of D are seen as counterclockwise oriented.*

The steps of the algorithm are intended, as in Alexander's theorem, to transform the diagram into a braided one. There are two main operations to be considered. The first one is the smoothing, cf. the picture below:



the smoothing (central) of the two types of crossings

The second is the bending:



a bending of two incompatible Seifert circles

Of course, a bending is in fact a Reidemeister II move, and so it does not change the link type. If we apply the smoothing operation at all crossings of D, we arrive at a disjoint union of what are called Seifert circles. Their number is denoted by $n(D)$.

Definition 2.5.3 *Two Seifert circles are called incompatible if when considered as closed curves in $\mathbb{S}^2$ they are oriented as the boundary of the (conveniently oriented) annulus they bound.*

Let $h(D)$ the number of pairs of incompatible Seifert circles. An important notion in the algorithm is the shadow $\mid D \mid$ of D:

Definition 2.5.4 *The shadow of D is the same diagram with all crossings replaced by simple intersections. It is a 4-valent graph denoted by $\mid D \mid$.*

Faces of $\mid D \mid$ are the connected components in $\mathbb{R}^2 \setminus \mid D \mid$.

Definition 2.5.5 *A face of $\mid D \mid$ is troubled if it has two opposite edges i.e. belonging to different and incompatible Seifert circles.*

The next two lemmas are the basis of the algorithm:

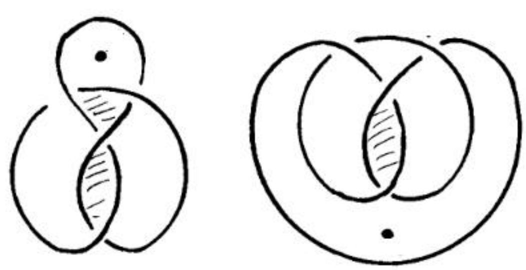
Lemma 2.5.6 *Let D' obtained from D by a bending along two opposite edges. Then $n(D') = n(D)$ and $h(D') = h(D) - 1$.*

Lemma 2.5.7 *The shadow of a link diagram D has a troubled face iff $h(D) > 0$.*

The next lines are the beginning of the algorithm (cf. Prasolov and Sossinsky [23] pp. 58):

DESTROY ALL CROSSINGS
WHILE THERE IS A TROUBLED REGION
DO A BENDING ALONG A PAIR OF OPPOSITE EDGES
DESTROY ALL CROSSINGS
END WHILE

Applying the above "program" we obtain a diagram without troubled regions. However it is not the end. The diagram is not necessarily braided. A last step is needed that can involve a so-called change of the infinity. (cf. figure below from [23] pp. 57)



change of the infinity

Lemma 2.5.8 *An oriented link diagram D in $\mathbb{R}^2$ with $h(D) = 0$, can be transformed using the Reidemeister II and III moves into a braided one.*

Remark 2.5.9 *In fact, the hypothesis* $h(D) = 0$ *implies that* D *viewed in* $\mathbb{S}^2$ *is isotopic with a diagram which is braided in* $\mathbb{R}^2$ *(cf. lemma 2.6 in [14]). The fact that isotopic diagrams in* $\mathbb{S}^2$ *represent equivalent links is 2.1.2 from [14]. The only subtle point here is when the isotopy crosses the infinity, and it can be proved that from the planar point of view this can be obtained by Reidemeister II and III moves.*

The above lemma gives the second part of the algorithm:

IF THE DIAGRAM IS BRAIDED
STOP
ELSE
DO CHANGE THE INFINITY
STOP.

In fact by the invariance of $n(D)$ under the bending operations and the fact that $n(D)$ is also invariant by isotopy, we conclude that finally we obtain a braid representation with $n(D)$ strands, the initial number of Seifert circles.

2.6 The fundamental group of the complement: braid group picture

We have seen in Introduction, that the way to define a computable invariant passes through the fundamental group of the link/knot complement. The present and the next section are devoted to two methods for calculating this group. Here we shall use a braid presentation for the link L. If σ is an n-braid with closure L and $\hat{\sigma}$ is the automorphism of the free group F_n in the letters $x_1, ..., x_n$ associated with σ, the following result due to Artin and Birman [22] gives a presentation of the fundamental group of the link complement:

Theorem 2.6.1 π *has a presentation of the form*

$$< x_1, ..., x_n \mid x_1 = \hat{\sigma}(x_1), ..., x_n = \hat{\sigma}(x_n) >.$$

Remark 2.6.2 *Any of the relations above is a consequence of the others and so we have a presentation with deficiency* 1*, a well known fact for link groups.*

As an application of the above theorem let's compute the Alexander polynomial of the Borromean link. The planar diagram is braided and a braid for the Borromean link is $(\sigma_2{}^{-1}\sigma_1)^3$. As consequence, the fundamental group π has the presentation:

$$\pi =< a, b, c \mid b = (\sigma_2{}^{-1}\sigma_1)^3 b, c = (\sigma_2{}^{-1}\sigma_1)^3 c >.$$

The relations can be written as $b = c^{-1}a^{-1}ca \cdot b \cdot a^{-1}c^{-1}ac$ and
$c = b^{-1}aba^{-1} \cdot c \cdot ab^{-1}a^{-1}b$.
With Fox calculus we obtain the following matrix M:

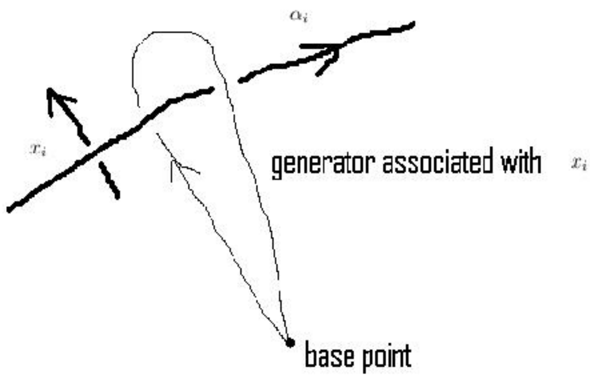
$$\begin{pmatrix} -(xyz)^{-1}(y-1)(z-1) & 0 & -(xyz)^{-1}(y-1)(x-1) \\ (xy)^{-1}(y-1)(z-1) & -(xy)^{-1}(y-1)(z-1) & 0 \end{pmatrix}$$

The elementary ideals are $E_1 = \mathbb{I}(\Delta)$ where $\Delta = (x-1)(y-1)(z-1)$ is the Alexander polynomial, and $E_2 = ((x-1)(y-1), (x-1)(z-1), (y-1)(z-1))$. The characteristic varieties are $\mathcal{V}_1 = V(\Delta) = \{(x, y, 1), (1, y, z), (x, 1, z)\}$ and $\mathcal{V}_2 = \{(x, 1, 1), (1, y, 1), (1, 1, z)\}$.

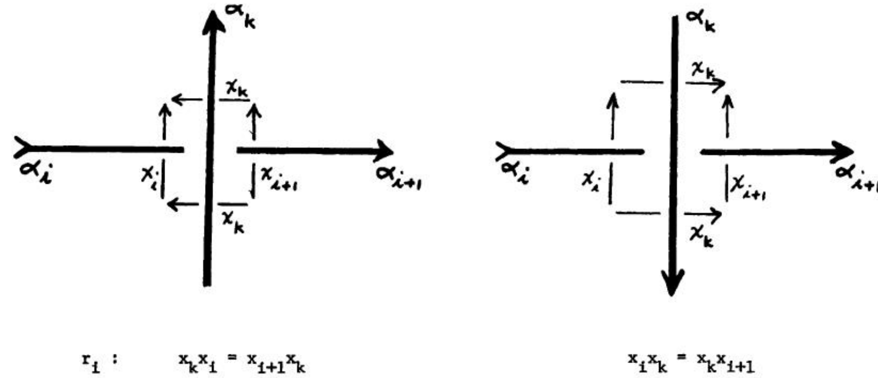
2.7 The fundamental group of the complement: Wirtinger picture

The aim of this section is to describe Wirtinger presentation for a knot complement following mainly Rolfsen [25]. For a knot K, we begin with a plane presentation and a chosen orientation on it. We shall denote every connected component by α_i and on it we shall chose an ortogonal vector x_i such that:
- it passes under α_i
- the "frame" α_i, x_i agrees with a chosen orientation of the plane.

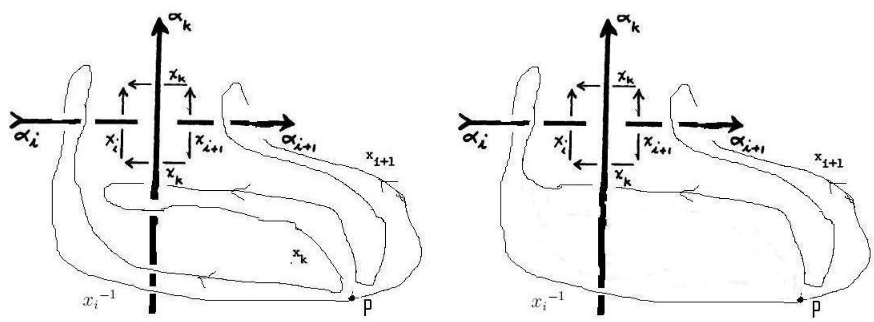
The $x_i's$ will be the generators of π in the sense that they represent based curves which go around the arcs $\alpha_i's$ cf. figure below:

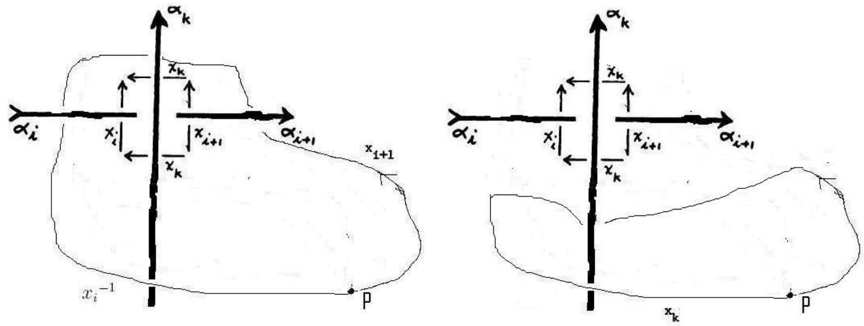


With respect to the types of crossings that can appear we can have the following situations: (figure from Rolfsen [25] pag. 57)



The following four steps show us the imposed relation $x_k = x_{i+1}x_kx_i{}^{-1}$for the left cross from above:

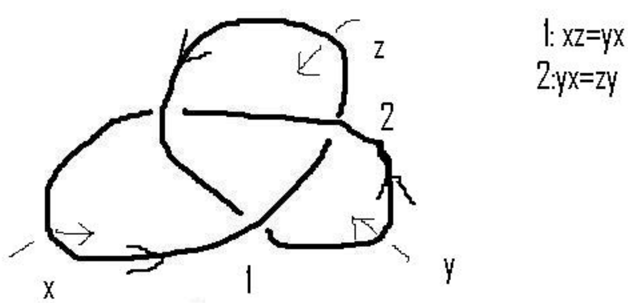




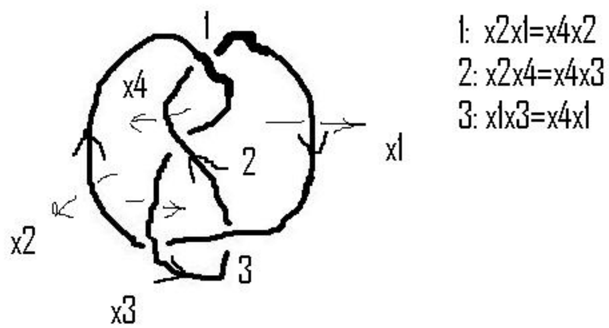
The algorithm above can be summarized in the following:

Theorem 2.7.1 *With the preceding notations* π *is presented with generators* $x_i's$ *and relations* $r_i's$, *where* $r_i = {x_k}^{-1}x_{i+1}x_k{x_i}^{-1}$ *or* $r_i = {x_k}^{-1}{x_i}^{-1}x_kx_{i+1}$. *Moreover, any relation is a consequence of the others, i.e. the presentation of* π *is with deficiency* 1.

Exemple 2.7.2 *For the left trefoil we have the following presentation* $\pi = < x, y \mid xyx = yxy >$.



Exemple 2.7.3 *For the eight knot we have the following presentation* $\pi = < x_1, x_2, x_3, x_4 \mid x_2x_1 = x_4x_2, x_2x_4 = x_4x_3, x_1x_3 = x_4x_1 >$.



the figure eight knot

Remark 2.7.4 *In fact the group of the figure eight knot can be presented by only two generators x, y with only one relation $yx^{-1}yxy^{-1} = x^{-1}yxy^{-1}x$.*

Chapter 3

Alexander polynomial of knots and links

The aim of this chapter, is to present several methods for the calculus of the Alexander polynomial for knots and links. As I mentioned in the Abstract, this invariant is the last, most tractable but coarser invariant on the road *homeo type of the complement* $-$ π *of the complement* $-$ *Alexander invariant* $-$*Alexander polynomial*. For a link L with q components in Σ an integral homology 3-sphere, we denote by $X := \Sigma \backslash L$ and $G := \pi_1(X)$. We have the following:

Proposition 3.0.1 *G is a finitely presented group with $def(G) > 0$.*

We recall that $\Delta_L(t_1, ..., t_q) \in \Lambda$ is the Alexander polynomial, A_L the Alexander module, $\mathcal{V}_k(L)$ are the Alexander varieties and from 1.3.2,

Corollary 3.0.2 *We have:*

$$E_1(A_L) = \begin{cases} (\Delta_K) & \text{if } q = 1 \\ (\Delta_L) \cdot \{\mathbb{I}\} & \text{if } q > 1 \end{cases}$$

Also, $\mathcal{V}_1(L) = V(\Delta_L) \subseteq \mathbb{T} = (\mathbb{C}^\star)^q$

3.1 Alexander polynomial via free Fox calculus

From one point of view, Fox calculus is a highly efficient method for the calculus of the Alexander polynomial starting from a presentation of the (fundamental) group. The main tool used is the notion of Fox derivation. Let G be a group, $\mathbb{Z}G$ its group ring and $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ the aditive agmentation morphism: $\epsilon(\Sigma n_i g_i) = \Sigma n_i$.

Definition 3.1.1 *A (Fox) derivation is a map $D : \mathbb{Z}G \to \mathbb{Z}G$ such that:*
i) is aditive
ii)$D(w_1 w_2) = D(w_1)\epsilon(w_2) + w_1 D(w_2)$.

A first interesting point is [3]:

Theorem 3.1.2 *For $G = F_n$, the free group on n letters $x_1, ..., x_n$, for any $1 \leq j \leq n$, there exist a unique derivation D_j such that $D_j(x_i) = \delta_i{}^j$.*

For a presentation of $G < x_1, ..., x_n \mid r_1, ..., r_m >$, we denote by ρ the composition

$$\mathbb{Z}F_n \to \mathbb{Z}G \to \mathbb{Z}G_{ab}.$$

Let's consider the $m \times n$ matrix $M_G := [\rho(D_j(r_i))]$ with entries from $\mathbb{Z}G_{ab}$. The central result for the Fox calculus is:

Fox Theorem 3.1.3 *M_G is a presentation matrix for the Alexander module A_G.*

Remark 3.1.4 *We notice that the matrix above is a presentation matrix for the Alexander module A_G and* **NOT** *for the Alexander invariant B_G.*

Let's compute the presentation matrix for A_G for two traditional examples from [27]:

Trefoil matrix 3.1.5 *A presentation matrix for the trefoil is*
$A = [t^2 - t + 1; -t^2 + t - 1]$, with t a generator of $G_{ab} = \mathbb{Z}$.

Proof: For G we have the presentation $< x, y \mid xyx = yxy >$.
$D_x(xyx) = 1 + xy$, $D_x(yxy) = y$ so $D_x(r) = 1 - y + xy$. Also, by the same method $D_y(r) = x - 1 - yx$. After taking the image by $\rho : \mathbb{Z}F_2 \to \mathbb{Z}G \to \mathbb{Z}\mathbb{Z}$, we obtain the mentioned result. QED

Remark 3.1.6 *In the calculus above we used the important fact that for a relation of the type $a = b$, for any Fox derivation, $D(ab^{-1}) = D(a) - D(b)$.*

Eight knot matrix 3.1.7 *A presentation matrix for the eight knot is* $A = [t - 3 + t^{-1}; -t + 3 - t^{-1}]$

Proof: As noted in 2.7.4 the fundamental group of the eight knot has a presentation of the form $< x, y \mid yx^{-1}yxy^{-1} = x^{-1}yxy^{-1}x >$. The images after abelianisation are:
$\rho(D_x(r)) = t - 3 + t^{-1}$ and $\rho(D_y(r)) = -t + 3 - t^{-1}$. QED

From the two examples above, we obtain the Alexander polynomial for the trefoil and figure eight knots:

$$\Delta_{trefoil} = t^2 - t + 1$$

$$\Delta_{fig\ eight} = t^2 - 3t + 1.$$

3.2 Alexander polynomial via braid groups

This section will provide a method to compute the Alexander polynomial of a link directly from an associated braid following Birman [1], Moran [22] and [14]. First of all we shall present what is named the Burau representation. Using it, there is a formula that produces the Alexander polynomial. For $n \geq 2$ and $i = 1...n-1$, we consider the $n \times n$ matrix U_i over the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$:

$$U_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}$$

A simple calculus show that these matrices satisfy

$$U_i U_j = U_j U_i \text{ for } \mid i - j \mid \geq 2$$

$$U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1} \text{ for } 1 \leq i \leq n-2.$$

and hence they produce a representation for the braid group B_n for $n \geq 2$ in $GL_n(\Lambda)$. It is the Burau representation denoted by ψ_n. By convention, for $n = 1$ one consider the trivial representation $B_1 \to GL_1(\Lambda)$. An important fact is that the Burau representations are compatible with the inclusions $i: B_n \subset B_{n+1}$, which means that for any $\beta \in B_n$ one has

$$\psi_{n+1}(i(\beta)) = \begin{pmatrix} \psi_n(\beta) & 0 \\ 0 & 1 \end{pmatrix}$$

Let's consider for $n \geq 3$ and $1 \leq i \leq n-1$ $(n-1) \times (n-1)$ matrices V_i defined by:

$$V_1 = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix},$$

$$V_{n-1} = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}$$

and for $1 < i < n-1$

$$V_i = \begin{pmatrix} I_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}.$$

Also, consider C the $n \times n$ matrix with 1 on and above the diagonal and 0, below. By $*_i$ we denote the row $1 \times (n-1)$ matrix with only 0's if $i < n-1$ or $(0, ..., 0, 1)$ if $i = n-1$. The Burau representations are in fact reducible ones and finally with the previous notations we can state the next theorem from [14]:

Theorem 3.2.1 *For $1 \leq i \leq n-1$ we have*

$$C^{-1} U_i C = \begin{pmatrix} V_i & 0 \\ \star_i & 1 \end{pmatrix}$$

As the U_i's verifies the braid relations, their conjugates by C verify the same relations and so we obtain what is called the reduced Burau representation ${\psi_n}^r : B_n \to GL_{n-1}(\Lambda)$. For $n = 2$ it is defined by sending σ_1 to the matrix $-t$.

With Markov theorem in mind consider the following:

Definition 3.2.2 *A sequence of mappings $f_n : B_n \to \mathbb{Z}[s, s^{-1}]$ is a Markov function if it is invariant under Markov moves.*

Remark 3.2.3 *In view of Markov theorem a Markov function produce a link invariant!*

Denote $g : \Lambda = \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[s, s^{-1}]$ the morphism which sends $t \to s^2$ and for $\beta \in B_n$, $< \beta >$ its image under the morphism $B_n \to \mathbb{Z}$ which sends all generators to 1. Also, consider the function $f_n : B_n \to \mathbb{Z}[s, s^{-1}]$ for $n \geq 2$

$$f_n(\beta) = a_n(\beta) \cdot g(det({\psi_n}^r(\beta) - I_{n-1})),$$

where, the multiplication factor a_n is defined by

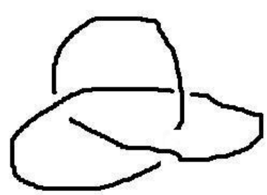
$$a_n(\beta) = (-1)^{n+1} \tfrac{s^{-<\beta>}(s-s^{-1})}{s^n - s^{-n}}.$$

For $n = 1$ we consider by definition $f_1(B_1) = 1$. We arrived at the followings two fundamental results:

Theorem 3.2.4 *The above f_n's defines a Markov function.*

Theorem 3.2.5 *For a link $L = \hat{\beta}$ for $\beta \in B_n$, $f_n(\beta)$ is the Alexander-Conway polynomial.*

Exemple 3.2.6 *For the right trefoil which is the closure of ${\sigma_1}^3 \in B_2$,*



right trefoil

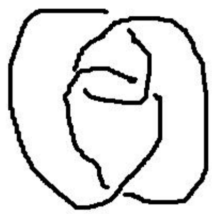
${\sigma_1}^3$

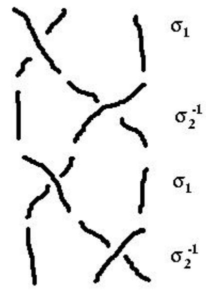
the above algorithm gives: $f_2({\sigma_1}^3) = s^2 + s^{-2} - 1$. For obtaining the normalized Alexander polynomial, we must consider the followings changes of variables:

$$s^{-1} - s \to \sqrt{t} - \sqrt{\tfrac{1}{t}}.$$

After this we arrive at the well known $t + t^{-1} - 1$.

Exemple 3.2.7 *For the figure eight knot depicted as below,*





the corresponding braid is $\sigma_1{\sigma_2}^{-1}\sigma_1{\sigma_2}^{-1} \in B_3$.

Using the two matrices

$$V_1 = \begin{pmatrix} -t & 0 \\ 1 & 1 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 1 & t \\ 0 & -t \end{pmatrix}$$

and $< \sigma_1{\sigma_2}^{-1}\sigma_1{\sigma_2}^{-1} >= 0$, we arrive at the well known $t + t^{-1} - 3$.

3.3 Alexander-Conway polynomial via skein relations

The present approach will produce an invariant for oriented links. At the end it will coincide with the previously defined versions of the Alexander invariant and in particular it is independent on the orientation in the knot case. We begin with the following:

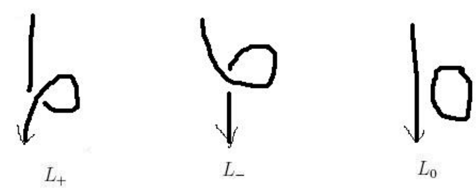
Definition 3.3.1 *A Conway triple is composed by* 3 *oriented links which, outside a ball coincide, while in the ball they look like in the figure below*

L_+ L_- L_0

Definition 3.3.2 *An Alexander Conway polynomial for links, is a function that assigns to every oriented link L, a polynomial $\nabla(L) \in \mathbb{Z}[s, s^{-1}]$ such that:*
1. $\nabla(L)$ is invariant under isotopy,
2. it is 1 for the trivial knot,
3. for any Conway triple, we have

$$\nabla(L_+) - \nabla(L_-) = (s^{-1} - s)\nabla(L_0).$$

Remark 3.3.3 *The example below*



is a Conway triple. It shows that if an Alexander Conway polynomial exists, it is 0 on all links which are union of a nonempty one with a trivial unlinked knot. In particular, on the trivial link with at last 2 components, it is 0.

The main result of this section is:

Theorem 3.3.4 *An Alexander Conway polynomial exists, is unique and coincides with the f_n defined using the reduced Burau representation.*

Proof: The proof, from [14], consists of two steps: we first prove uniqueness, and then, we show that the previously defined invariant satisfies the skein relation.
Uniqueness: we need the following

Definition 3.3.5 *An oriented link diagram D is ascending, if it satisfies:*
1. the link components can be indexed such that at every cross, the component with smaller index goes below that with a greater one,
2. any component has a base point (not a cross), such that if one move in the positive direction from that point, we meet every self-crossing first on the undergoing branch.

An ascending diagram represents always the trivial link.
Suppose that we have two Alexander Conway functions and let ∇ their difference. We shall prove that $\nabla = 0$.
∇ is 0 on trivial knots and links and verify the skein relation. We proceed by induction on N-the number of crossings. For $N = 0$, we have a trivial knot/link an so the induction starts. Suppose it is true at level N. Let L a link presented with $N + 1$ crossings. At a cross, if we try to apply the skein relation, we obtain two other links: one , L' with the same number of crossings but the other, L_0-the smoothed one, with only N crossings. By induction, $\nabla(L_0) = 0$ and so, ∇ is unchanged if we change any cross in L. But with these moves, we always reach the trivial link. So $\nabla(L) = 0$ as we desired.
Existence: we know that the f_n defines a Markov function and hence a link invariant (under isotopy). It is easy that it is 1 on the trivial knot. We need to show the skein relation. Let $n \geq 2$, $1 \leq i \leq n - 1$ and α, β two braids in B_n. We have the following:
FACT: $\alpha\sigma_i\beta$, $\alpha{\sigma_i}^{-1}\beta$ and $\alpha\beta$ are a Conway triple.
The main point is that the proof of Alexander theorem (any link is the closure of a braid) show that any Conway triple is of this type. So we need to prove the skein relation only for triples as above:

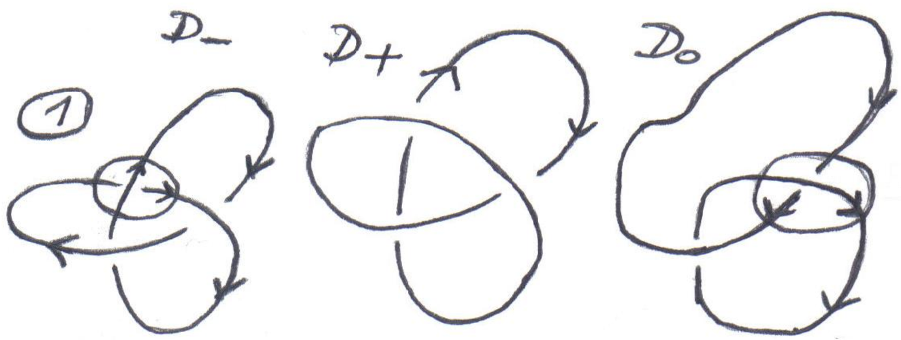
$$f_n(\alpha\sigma_i\beta) - f_n(\alpha{\sigma_i}^{-1}\beta) = (s^{-1} - s)f_n(\alpha\beta).$$

But f_n is invariant under conjugation and σ_i is conjugated with σ_1, so we may assume $i = 1$. Also, by conjugation with α we can assume $\alpha = 1$. So we need to prove:

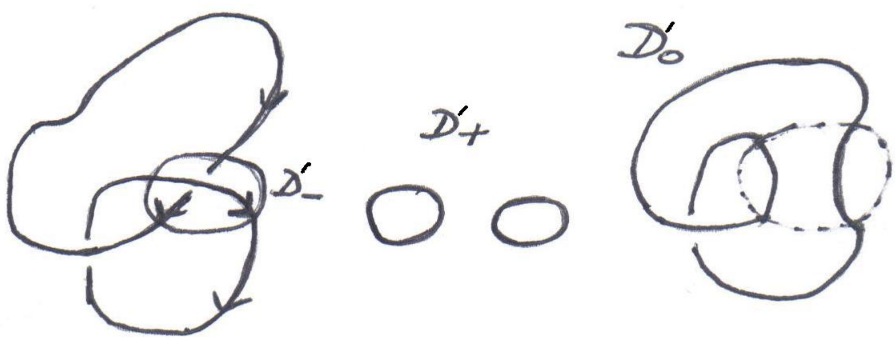
$$f_n(\sigma_1\beta) - f_n({\sigma_1}^{-1}\beta) = (s^{-1} - s)f_n(\beta).$$

An intricate calculus, using the explicit form of f_n (page. 117 from [14]) show the above relation. A last point to be verified is that the Alexander Conway invariant take values in Laurent polynomials, but it is an easy induction on the number of crossings that it is a polynomial in $s^{-1} - s$. **QED**

Exemple 3.3.6 *We apply this method to our favorite link: the left trefoil.*



For the triple above, $\nabla(D_-) = \nabla(D_+) - z\nabla(D_0)$*, where by* z *we denote the* $s^{-1} - s$ *from above.*
In the next step below, D_0 *became* $D_-{}'$



and we have: $\nabla(D_0) = \nabla(D_-{}') = \nabla(D_+{}') - z\nabla(D_0{}') = -z$*. So, the Alexander for the trefoil is:* $\Delta(t) = \nabla(\sqrt{t} - \sqrt{\frac{1}{t}}) = 1 + (\sqrt{t} - \sqrt{\frac{1}{t}})^2 = t + t^{-1} - 1$.

3.4 General properties of the Alexander polynomial

This section collects the main properties of the Alexander polynomial for knots and links cf. [2] [18] [25].

KNOTS

Theorem 3.4.1 *For a knot* K*,* $\Delta_K(t)$ *is symmetric (up to multiplication by units).*

Theorem 3.4.2 *For a knot,* $\Delta_K(1) = 1$.

If we denote by $-K$ the same knot with orientation reversed and by K^* the mirror image (for a plane diagram this means changing all crossings) we have:

Theorem 3.4.3 $\Delta_{-K} = \Delta_K$ *and* $\Delta_{K^*} = \Delta_K$.

In particular, the Alexander polynomial does not distinguish knots from their mirror images.
Also, for factorizable knots we have:

Theorem 3.4.4 $\Delta_{K_1 \sharp K_2} = \Delta_{K_1} \cdot \Delta_{K_2}$.

LINKS

We recall that a link is splittable if there is an embedded 2-sphere disjoint from L which separates some components of L from the others. A first property is:

Theorem 3.4.5 *For a splittable link L we have* $\Delta_L = 0$.

Theorem 3.4.6 *For L a link with r components, the r-variables Alexander polynomial verifies the following Torres relations (up to multiplication by units):*

$$\Delta_L(t_1, ..., t_r) = \Delta_L({t_1}^{-1}, ..., {t_r}^{-1})$$

and

$$\Delta_L(t_1, ..., t_{r-1}, 1) = \begin{cases} \frac{{t_1}^{l_1} - 1}{t_1 - 1} \Delta_{L'}(t_1), & \textit{if } r = 2 \\ {t_1}^{l_1} ... {t_{r-1}}^{l_{r-1}} \Delta_{L'}(t_1, ..., t_{r-1}), & \textit{if } r \geq 3 \end{cases}$$

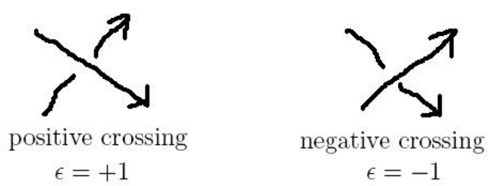
where $L = K_1 \cup ... \cup K_r$, $L' = K_1 \cup ... \cup K_{r-1}$ *and* $l_i = lk(K_i, K_r)$.

Chapter 4

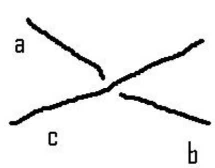
Applications

4.1 Fox coloring

The references for this section are [18], [24] and [26]. For an oriented link $L \subset \mathbb{R}^3$, a planar diagram for it in $\mathbb{R}^2$ will be denoted by D. In general a Q-coloring of D is a map from the arcs of D (from the set of its connected components) to the set of "colors" Q, such that certain conditions are satisfied at each cross. The crosses are defined to be positive/negative as in the following picture:



Definition 4.1.1 *A Fox or n-coloring is a $\mathbb{Z}_n$-coloring, with at least 2 colors such that at a colored cross*



we have: $a + b = 2c$ (mod n).

Let $Col_n(D) = \{n\text{-colorings of } D\}$ and $c_n(D) = \mid Col_n(D) \mid =$ number of n-colorings. With these notations we have the following results from [24] and [2]:

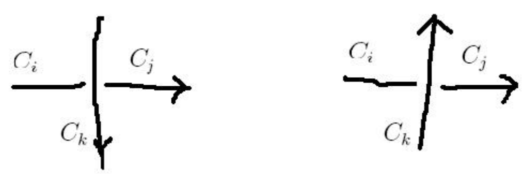
Proposition 4.1.2 *$Col_n(D)$ is an abelian group, $c_n(D)$ is a link invariant and for $n = p$, a prime number, $c_p(D)$ is a p-power, p^ν.*

Proposition 4.1.3 *There is a bijection between $Col_n(D) \longleftrightarrow Hom(G, D_{2n})$ such that for $c \in Col_n(D)$ with $c(s_i) = a_i$ the corresponding homomorphism is the one which sends $s_i \to a_i b$.*

Theorem 4.1.4 *If $L = K$ is a knot and $n = p$ a prime number, then K has a p-coloring iff $\Delta_K(-1) \equiv 0 (mod\ p)$.*

Let $n, d \geq 2$ integers, $Q = {\mathbb{Z}_n}^d$ and Φ the companion matrix of the degree d cyclotomic polynomial.

Definition 4.1.5 *A generalized Fox (or (n, d)) coloring is a Q-coloring such that, at every cross as below,*



$\epsilon = +1$ $\epsilon = -1$

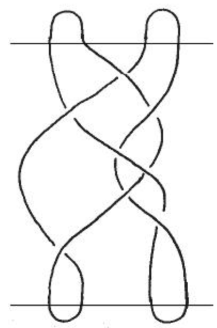
the following relation holds: $(C_i - C_k)\Phi^\epsilon = C_j - C_k$.

In the above setting, $Col_{n,d}(D)$ is a $\mathbb{Z}_n$-module.
More generally, for Q an abelian group and $\Phi \in Aut(Q)$ a fixed automorphism, we can define by the same method a generalized Fox Q-coloring by imposing the same relation at the crossings: $(C_i - C_k)\Phi^\epsilon = C_j - C_k$.

4.2 2-Bridge knots and links

In this section we will discuss about a particular class of knots/links named 2-bridge.

Definition 4.2.1 *A 2-bridge link is one which has a planar diagram with exactly 2 minimums and 2 maximums. More precisely it is a particular type of closure of a 4-braid as in the picture bellow (cf. [2] pp. 25).*



These links are classified by 2 numbers (α, β) and denoted $K(\alpha, \beta)$ as in the following:

Theorem 4.2.2 *2-Bridge links are determined by a pair (α, β) such that:*
1. $0 < \beta < \alpha$, *β is odd and $gcd(\alpha, \beta) = 1$*
2. the 4-braid associated to the link is

$$\sigma_1{}^{a_1}\sigma_2{}^{-a_2}...\sigma_1{}^{a_m}$$

where m is odd and $[a_1, ..., a_m]$ are the quotients of the continued faction of $\frac{\beta}{\alpha}$.
3. $K(\alpha, \beta)$ is a knot for α odd and a link with 2 components for α even.

Exemple 4.2.3 *1. $K(3.1)$ is the trefoil.*
2. $K(\alpha, 1)$ is the $(2, \alpha)$ torus knot.
3. $K(5, 3)$ is the figure eight knot.
4. $K(8, 5)$ is the Whitehead link.

The next step is to describe the fundamental group of $K(\alpha, \beta)$. For this we introduce:

$$\epsilon_i = (-1)^{[\frac{i \cdot \beta}{\alpha}]} \text{ for } i = 1, ..., \alpha - 1.$$

For a and b free variables we denote $w = b^{\epsilon_1} a^{\epsilon_2} ... a^{\epsilon_{\alpha-1}}$ if α is odd and $w' = b^{\epsilon_1} a^{\epsilon_2} ... b^{\epsilon_{\alpha-1}}$ if α is even. We have the following theorem from [7]:

Theorem 4.2.4 *If α is odd, the fundamental group of the complement of the knot $K(\alpha, \beta)$ has the following presentation:*

$$G(\alpha, \beta) =< a, b \mid aw = wb >.$$

If α is even the fundamental group of the 2-component link $K(\alpha, \beta)$ has the following presentation:

$$G(\alpha, \beta) =< a, b \mid aw' = w'a >.$$

The following theorem from [13] concerns the Alexander Polynomial of 2-bridge knot:

Theorem 4.2.5 *The Alexander polynomial of the knot $K(\alpha, \beta)$ is:*

$$\Delta_{K(\alpha,\beta)}(t) = 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - ... + t^{\epsilon_1+...+\epsilon_{\alpha-1}}$$

Remark 4.2.6 *For 2-bridge knots we have the following formula for the determinant: $\mid \Delta_{K(\alpha,\beta)}(-1) \mid= \alpha$.*

Proof: Using the formula for the Alexander polynomial in the previous theorem and using the fact that the *mod* 2 class of the power of t alternates we obtain $+1$ for each term in the sum. **QED**

For two bridge links we have:

Theorem 4.2.7 *For α even, the linking number of the two components of $K(\alpha, \beta)$ is:*

$$lk(K_1, K_2) = \sum_{j=1}^{\frac{\alpha}{2}} \epsilon_{2j-1}.$$

Also, using Torres relations we obtain:

Theorem 4.2.8 *Let $l = lk(K_1, K_2)$, the linking number of the two components in $K(\alpha, \beta)$. Then:*

$$\Delta_{K(\alpha,\beta)}(-1, 1) = \Delta_{K(\alpha,\beta)}(1, -1) = \tfrac{1-(-1)^l}{2}.$$

4.3 Metabelian quotients

In this section we consider metabelian representation of knots groups. The main references are [18] and [11].

Definition 4.3.1 *A group Γ is called metabelian if it is an extension of two abelian groups:*

$$0 \to A \to \Gamma \to B \to 0$$

The last arrow is denoted by $\pi : \Gamma \to B$. We shall assume that Γ is in fact a semi-direct product $A \rtimes B$. As a set Γ is $A \times B$, but there is a morphism $\alpha : B \to Aut(A)$, $b \to \alpha_b$ in terms of which, the multiplication in Γ is given by:

$$(a, b)(a', b') = (a + \alpha_b(a'), bb').$$

An equivalent formulation is that π has a section π' whose composition is the identity on B. Below are some examples:

Exemple 4.3.2 *1) $\Gamma = D_{2n}$, the dihedral groups $\mathbb{Z}_n \rtimes \mathbb{Z}_2$.*

$$1 \to \mathbb{Z}_n =< a > \to \Gamma \to \mathbb{Z}_2 =< b > \to 1$$

where the the $\mathbb{Z}_2$-action α on $\mathbb{Z}_n$ is given by $-1 \to \{a \to -a\}$. In fact Γ has the following presentation:

$$\Gamma =< a,\ b \mid a^n = b^2 = 1,\ bab = a^{-1} >.$$

2) The metacyclic groups $\Gamma = \mathbb{Z}_n \rtimes \mathbb{Z}_m$, where m is a divisor of the order of $Aut(\mathbb{Z}_n)$.
3) $\Gamma = {\mathbb{Z}_n}^d \rtimes \mathbb{Z}_m$ with m a divisor of $\mid Aut({\mathbb{Z}_n}^d) \mid$. For example the A_4 group is ${\mathbb{Z}_2}^2 \rtimes \mathbb{Z}_3$.

For a group G we shall consider the set $Rep(G, \Gamma)$. To obtain a description of it, we fix a morphism $\mu : G \to B$ and we try to understand the set $\{\lambda : G \to \Gamma \mid \pi \circ \lambda = \mu\}$, denoted by $Rep_\mu(G, \Gamma)$.

Definition 4.3.3 *A derivation $\phi : G \to A$ is a map such that $\phi(gh) = \phi(g) + g \cdot \phi(h)$, where the last multiplication, the G-module structure on A is given by $\rho = \alpha \circ \mu$. Also, we denote by ${Z_\rho}^1(G, A)$ the set of derivations.*

The description of $Rep_\mu(G,\Gamma)$, is given by the following:

Proposition 4.3.4 *There are bijections*

$$Rep_\mu(G,\Gamma) \longleftrightarrow {Z_\rho}^1(G,A)$$

and

$$Rep_\mu(G,\Gamma)/\{inner\ auto's\ of\ \Gamma\ with\ el'ts\ in\ A\} \longleftrightarrow {H_\rho}^1(G,A).$$

The problem of the existence of morphisms from knot groups to metabelian groups is a difficult one. The following theorem, due to Fox, gives a criterion for the existence of such a morphism in the dihedral case, in terms of the Alexander polynomial of the knot:

Theorem 4.3.5 *For a knot K, there is a nontrivial morphism $\pi_1(K) \to D_{2p}$ with p prime iff*

$$\Delta_K(-1) \equiv 0\ mod\ p.$$

An extension of the above theorem, was obtained by Matei and Suciu [19] in the following setting: for p, q prime numbers, denote by s the order of q *mod* p in ${\mathbb{Z}_p}^\star$. The next lemma (pp. 485 in [19]) describes the metacyclic extensions M_{p,q^s}:

Lemma 4.3.6 *1) There is an automorphism $\sigma \in Aut({\mathbb{Z}_q}^s)$ of order p.*
2) All these automorphisms give isomorphic metacyclic extensions.
3) $Aut({\mathbb{Z}_q}^s \rtimes_\sigma \mathbb{Z}_p)$ has $sq^s(q^s-1)$ elements.

For G a finitely generated group, $\mathbb{K}$ a field and $t : G \to \mathbb{K}^\star$ a character, the depth of t is:

$$d_{\mathbb{K}}(t) = max\{d \mid t \in \mathcal{V}_d(G,\mathbb{K})\}.$$

Remark 4.3.7 *We have $0 \le d_{\mathbb{K}}(t) \le l(G)$ (cf. [19] pp. 481), where $l(G)$ is the minimal number of generators in a finite presentation of G.*

Let b the generator of $\mathbb{Z}_p$ and ${\mathbb{Z}_q}^s$ viewed as the additive group of $\mathbb{K} = \mathbb{F}_q(\xi)$, where $\xi \in \mathbb{K}^\star$ is a primitive p-th rooth. Then $\sigma(b)$ can be identified with $\xi \in Aut(\mathbb{K})$ and $\mathbb{Z}_p$ with a sub-group in $\mathbb{K}^\star$. So, $Hom(G,\mathbb{Z}_p)$ is a sub-set in the character torus $Hom(G,\mathbb{K}^\star)$. The next theorem (pp. 487 in [19]) computes the number of (epi)morphisms from G to M_{p,q^s}:

Theorem 4.3.8 *The number of homomorphisms is:*

$$| \, Hom(G, M_{p,q^s}) \, |= \Sigma \; q^{sd_{\mathbb{K}}(\rho)+s},$$

the sum being after all $\rho \in Hom(G, \mathbb{Z}_p)$.
The number of epimorphisms is

$$| \, Epi(G, M_{p,q^s}) \, |= \Sigma \; q^s(q^{sd_{\mathbb{K}}(\rho)} - 1),$$

the sum being over all non-trivial ρ.

Another interesting problem for a finitely generated group G and a finite one Γ is to compute the Hall invariants:

$$\delta_\Gamma(G) :=| \, Epi(G, \Gamma)/Aut\Gamma \, |.$$

In the above metacyclic setting we define

$$\mathrm{Tors}_{p,d}(G, \mathbb{K}) = \{\rho \in \mathcal{V}_d(G, \mathbb{K}) \text{ of order exactly } p\}.$$

The result below (cf. pp. 488 and 483 in [19]) compute the Hall invariants in terms of the number of torsion points in characteristic varieties of M_{p,q^s}.

Theorem 4.3.9 *The Hall invariants are:*

$$\delta_{M_{p,q^s}}(G) = \tfrac{p-1}{s(q^s-1)} \cdot \Sigma \; {\beta_{p,d}}^{(q)}(G) \cdot (q^{sd} - 1)$$

the sum being over $d \geq 1$ and

$${\beta_{p,d}}^{(q)}(G) = \tfrac{1}{p-1} \cdot | \; Tors_{\;p,d}(G, \mathbb{K}) \backslash \; Tors_{\;p,d+1}(G, \mathbb{K}) \, |.$$

As an application we will compute the number of epimorphisms between the fundamental group of a 2-bridge knot and a dihedral group D_{2m}. We know that $G(\alpha, \beta) =< x, y \mid xw = wy >$,
$D_{2m} =< a, b \mid a^m = 1; b^2 = 1; bab = a^{-1} >$ and we have the following exact sequence:

$$1 \to \mathbb{Z}_m \to D_{2m} \to \mathbb{Z}_2 \to 1.$$

A homomorphism $f \in Hom(G, D_{2m})$ is determined by (u, v, ϵ) such that $f(x) = a^u b^\epsilon$, $f(y) = a^v b^\epsilon$.
Denote by $\Psi : G \to \mathbb{Z}_m$, $\Psi(x) = u$ and $\Psi(y) = v$, $M = [(\frac{\partial r_i}{\partial x_j})]$ the Alexander matrix and $\rho : G \to \mathbb{Z}_2$ the character such that the induced $\rho' : G_{ab} = \mathbb{Z} \to \mathbb{Z}_2$ is $\rho'(1) = (-1)^\epsilon$.
We have $f \in Hom(G, D_{2m})$ iff $\Psi \in Der_\rho(G, \mathbb{Z}_m)$. The last condition is equivalent with: $M((-1)^\epsilon)(u, v) \equiv \ 0 \ (mod \ m)$.
Recall that the Alexander matrix for 2-bridge knot is $M(t) = [-\Delta(t) \ \Delta(t)]$ where Δ is the Alexander polynomial.
Denote by $\Phi_H(G) =\mid Epi(G \to H) \mid$ and $\sigma_H(G) =\mid Hom(G \to H) \mid$.
So the condition for f to be a homomorphism is equivalent to the following equation: $\Delta((-1)^\epsilon) \cdot (u - v) \equiv \ 0 \ (mod \ m)$.
Case1 If $\epsilon = 0 \Rightarrow Imf \sim \mathbb{Z}_l \subseteq \mathbb{Z}_m$ with $l \mid m$.
But $\Phi_{\mathbb{Z}_l}(G) = \Phi_{\mathbb{Z}_l}(\mathbb{Z}) = \phi(l)$ the last term being the Euler function. We know the formula:

$$\sum_{l|m} \phi(l) = m.$$

So, in this case we obtain m homomorphisms.
Case2 If $\epsilon = 1$, we have from the properties of Alexander polynomial for 2-bridge knots that $\Delta(-1) = \alpha$, so the equation: $\alpha \cdot z \equiv \ 0 \ (mod \ m)$ where $z := u - v$.
Denote by $d = gcd(\alpha, m)$, $\alpha = d\alpha'$, $m = dm'$ with $gcd(\alpha', m') = 1 \Rightarrow z \in \{m', 2m', ..., dm'\}$ so $gcd(\alpha, m)$ solutions.
For each solution z, we have $u = z + v$, $v \in \mathbb{Z}_m$ so we obtain m pairs (u, v). It means that there are $m \cdot gcd(\alpha, m)$ homomorphisms.
From **Case1** and **Case2** we conclude that:

$$\sigma_{D_{2m}} = \sigma_{D_{2m}} G(\alpha, \beta) = m \cdot gcd(\alpha, m) + m.$$

The dihedral group D_{2m} has one subgroup which is isomorphic with $\mathbb{Z}_l$ and $\frac{m}{l}$ subgroups isomorphic with D_{2l}, for $l \mid m$.
We will express the number of homomorphisms by the number of epimorphisms as follows:

$$\sigma_{D_{2m}} = \sum_{l|m} \left[\frac{m}{l} \Phi_{D_{2l}} + \Phi_{\mathbb{Z}_l}\right].$$

Applying Moebius inversion

$$\Phi_{D_{2m}} = \sum_{l|m} \tfrac{m}{l}\mu(\tfrac{m}{l})[\sigma_{D_{2l}} - \sigma_{\mathbb{Z}_l}] =$$

$$= \sum_{l|m} \tfrac{m}{l}\mu(\tfrac{m}{l})[l \cdot gcd(\alpha, l) + l - l] =$$

$$= m \cdot \sum_{l|m} \mu(\tfrac{m}{l}) gcd(\alpha, l).$$

After some computations it follows that:

$$\Phi_{D_{2m}} G(\alpha, \beta) = \begin{cases} m \cdot \phi(m) & \text{if } \alpha \equiv 1 (mod\ m) \\ 0 & otherwise \end{cases}$$

The Hall invariant for 2-bridge knot $K(\alpha, \beta)$ is:

$$\delta_{D_{2m}} = \tfrac{\Phi_{D_{2m}}}{|Aut D_{2m}|} = \begin{cases} 1 & \text{if } \alpha \equiv 1 (mod\ m) \\ 0 & otherwise \end{cases}$$

Appendix

A1. Manifolds and duality

The main reference for this section is Hatcher [10].

Definition *An n-manifold M is a Hausdorff topological space, where every point has a neighborhood homeomorphic with $\mathbb{R}^n$.*

A compact manifold is named closed.
As first examples of manifolds we have $\mathbb{R}^n$, the spheres, the real or complex projective spaces, the open Moebius band, the genus g surfaces and the Klein bottle.
For any commutative with unity ring R (usually it will be $\mathbb{Z}$ or $\mathbb{Z}_2$) and any point $x \in M$ we have $H_n(M, M \setminus x, R) \simeq R$ (by excision and homology of S^{n-1}).
An orientation of M is a function which for each $x \in M$ assign a generator e_x of $H_n(M, M \setminus x, R)$, such that the following hold:

Compatibility condition *For any x there exist a chart neighborhood U and a generator e_U of $H_n(M, M \setminus U, R) \simeq R$ which for every $y \in U$ goes over e_y by the natural map $H_n(M, M \setminus U, R) \to H_n(M, M \setminus y, R)$.*

A first observation to made is that an orientation need not exists. An orientable manifold is one for which an orientation exists. For example on the Moebius band or the Klein bottle there is no orientation, but the spheres, any complex manifold and the real projective spaces of odd dimension are orientable. However any manifold admits a two-sheeted orientable covering.
For closed orientable and connected n-manifolds we have the following:

Theorem *The natural map*

$$H_n(M,R) \to H_n(M, M \setminus x, R)$$

is an isomorphism for all $x \in M$.

Under the above conditions a generator of $H_n(M,R)$ is named a fundamental or orientation class. For going further towards the Poincare duality theorem, we recall that for any space X there is a cap product defined at the chain-cochain level,

$$\cap : C_k(X,R) \times C^l(X,R) \to C_{k-l}(X,R),$$

inducing a well defined, R-linear in each argument, map denoted also by $\cap$:

$$H_k(X,R) \times H^l(X,R) \to H_{k-l}(X,R).$$

For $\sigma : \triangle^k \to X$ and $\varphi \in C^l(X,R)$, $\sigma \cap \varphi$ is defined by:

$$\varphi(\sigma(v_0,...,v_l))\sigma(v_l,...,v_k),$$

where $v_0,...,v_k$ are the vertices of the standard k-simplex. With the above in mind, we arrive at the famous:

Poincare duality theorem *For M closed orientable with chosen fundamental class $\mu \in H_n(M,R)$, the map $PD : H^k(M,R) \to H_{n-k}(M,R)$ defined by $PD(x) = \mu \cap x$ is an isomorphism for every k.*

For orientable noncompact manifolds there is no fundamental class; however using cohomology with compact support there is a version of Poincare duality morphism ${H^k}_c(M,R) \to H_{n-k}(M,R)$ which is still an isomorphism. Another form of generalization of the Poincare duality is for manifolds with boundary.

Definition *An n-manifold M is a Hausdorff topological space, where every point has a neighborhood homeomorphic with $\mathbb{R}^n$ or with ${\mathbb{R}^n}_+$(the closed upper half space determined by the last coordinate).*

Points which by the chart homeomorphism go to a point with $x_n = 0$ forms the boundary ∂M (an $n-1$ manifold). A manifold with boundary is by definition orientable if $M \setminus \partial M$ is. For a compact one, there is a (fundamental) class μ in $H_n(M, \partial M, R)$ which restricts to the chosen orientation class at every point in $M \setminus \partial M$. Using product with μ we have:

Poincare-Lefschetz duality *Let M compact orientable n-manifold with his boundary decomposed as $A \cup B$, where A, B are $n-1$-manifolds with common boundary $A \cap B$. Then,*

$$PD : H^k(M, A, R) \to H_{n-k}(M, B, R)$$

is an isomorphism.

In particular: for $A = \Phi$ and $B = \partial M$,

$$H^k(M, R) \simeq H_{n-k}(M, \partial M, R);$$

for $B = \Phi$ and $A = \partial M$,

$$H^k(M, \partial M, R) \simeq H_{n-k}(M, R).$$

A2. Immersions, embeddings, isotopy

In the preceding section manifolds were only topological. In this one, sometimes a manifold will be differentiable which means that it has an open covering with chart domains, such that the transition functions are differentiable of class C^∞.

Definition *A differentiable map $f : M \to N$ between smooth manifolds is an immersion if at every point $x \in M$ the differential is injective.*

Definition *A continous map $f : M \to N$ between topological spaces is a topological embedding if it is injective and homeo on its image.(the image having the subspace topology)*

Note that an injective map needs not always be a topological embedding; for example the figure eight in the plane is the injective image of any open interval of $\mathbb{R}$, without being homeo with it. However we have the following easy result:

Theorem *An injective map from a compact space into a Hausdorff one is a topological embedding.*

In the differentiable case we have:

Definition *A differentiable map $f : M \to N$ between smooth manifolds is a differentiable embedding if it is an immersion and a topological embedding.*

We remark that there are differentiable topological embeddings which are not differentiable embeddings:$f(x) = x^3$ from $\mathbb{R} \to \mathbb{R}$ is an obvious example. The third subject of this section is the notion of isotopy.

Definition *Two topological embeddings $f, g : X \to Y$ are isotopic if they are homotopic through embeddings (i.e. at each floor the corresponding map is an embedding).*

A related notion is the ambient isotopy:

Definition *Two topological embeddings $f, g : X \to Y$ are ambient isotopic if there exists $F : Y \times [0, 1] \to Y$ such that $F_0 = id$, F_t is a homeo and $F_1 \circ f = g$.*

It is clear that ambient isotopy is a stronger relation than isotopy and in fact it is the equivalence we are working with in knot/link theory. However in the differentiable setting with compact source these are in fact equivalent cf. Hirsch [12]. If the source is not compact one can consider a line in $\mathbb{R}^3$ and a line modified somewhere by a trefoil knot. These are isotopic but not ambient isotopic because their complements have not the same π_1. For knots/links, there is also the following third notion introduced by Kauffman [15], but we will not be concerned with it in this thesis.

Definition *Two link diagrams are regular isotopic if they are connected through type II and III Reidemeister moves.*

A3. Rings and orders

The aim of this section is to describe the theory of orders and Fitting ideals of modules, using as main references [18], [4], [6], [22], [25] and [27]. R will always denote a commutative, integer, unitary, unique factorization ring (UFD for short). Sometimes it will be even a principal ideal domain (PID for short). M is a finitely presented R-module. A presentation of M is an exact sequence

$$R^m \to R^n \to M \to 0$$

where m, n positive integers. Using standard basis in the free modules above, the presentation is encoded in an $m \times n$ matrix A with elements from R. We have the following fundamental definition:

Definition *For natural $k \geq 0$ the k-elementary ideal (or Fitting or determinantal) is the ideal in R denoted by $E_k(M)$generated by all $(n-k) \times (n-k)$ minors from A.(by convention they are 0 if $n-k > m$ and all R if $n-k \leq 0$)*

As far as every minor is a linear combination of sub-minors, the elementary ideals forms an ascending sequence. We have the following theorem:

Theorem *The elementary ideals are invariants of the module M and do not depend on the chosen presentation.*

Now, R being an UFD and M finitely presented, if $\Delta_k(M)$ is the greatest common divisor of elements in $E_k(M)$, then $\Delta_0(M)$ is called the order of M and we have:

Lemma $\Delta_{k+1}(M) \mid \Delta_k(M)$ *for $k \geq 0$.*

Exemple *Suppose $M = R^r \oplus \frac{R}{(p_1)} \oplus ... \oplus \frac{R}{(p_s)}$ is a finitely generated module over a PID (eg. $R = \mathbb{K}[t^{\pm 1}]$, where $\mathbb{K}$ is a field). Then:*

$$\Delta_i(M) = \begin{cases} 0 & \text{if } i < r \\ 1 & \text{if } i \geq r+s \\ p_{i-r+1}...p_s & \text{if } r \leq i < r+s \end{cases}$$

Remark *Through this thesis, R will be $\mathbb{K}[{t_1}^{\pm 1}, ..., {t_q}^{\pm 1}]$, for $\mathbb{K} = \mathbb{Z}$ or $\mathbb{C}$, the Laurent polynomials ring; it will be denoted by $\Lambda \otimes \mathbb{K}$.*

For E an ideal in R as above, $V(E)$ is the reduced variety defined by E in $S = Spec(R)$. For M a finitely generated R-module

$$supp(M) := V(ord\ M).$$

Moreover, we have the following:

Definition *The k^{th}-support variety of M is*

$$V_k(M) := V(E_{k-1}(M)) \subseteq S.$$

In particular $V_1(M) = V(ord\ M)$.

Exemple *For* $R = \Lambda \otimes \mathbb{C} = \mathbb{C}[{t_1}^{\pm 1}, ..., {t_q}^{\pm 1}],\ S = Spec(R) = (\mathbb{C}^\star)^q =: \mathbb{T}$.

For an ideal $E \neq 0$ in $\Lambda \otimes \mathbb{C}$ and $\Delta = gcd(E)$ the generator of the smallest principal ideal containing E, $\mathbb{T} \supseteq V(E) \supseteq V(\Delta)$ and we have:

Lemma $V(\Delta) = W_1(E) :=$ *the union of all codimension* 1 *irreducible components of* $V(E)$.

Directions for further study

As we had seen along the above pages, the Alexander modules/polynomials are strong tools for the study of topological and algebraic properties of 3-dimensional complements. For the future, I think that many directions are possible. First of all, a further study concerning the relation between Alexander varieties and metabelian representations should be very interesting.

Secondly, the modern direction toward the "twisted" world would be a natural next step.

Last but not least, an interesting route is the study of the Alexander ideas in other contexts like complements of projective hyper-surfaces.

All these directions are under current development across the world.

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