UNIVERSITY PARIS DIDEROT
DEPARTMENT OF MATHEMATICS

MASTER THESIS

RENORMALIZED QUANTUM DIMENSION AND MULTIVARIABLE LINK INVARIANTS

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Introduction

In this Thesis, we will study a family of multivariable invariants introduced by Nathan Geer and Bertrand Patureau in 2010 in the paper "Multivariable link invariants arising from super Lie algebras of type I". The starting point in this construction is the method defined by Reshetikhin-Turaev. In 1991, they described one construction which has as an input a Ribbon Category and obtains invariants for colored tangles. In particular, looking at a link as a endomorphism of the empty set, it gives link invariants. In this definition, an important role is played by the so called quantum dimension of an object. People often applied this construction for Ribbon categories which come from the representation theory of some Hopf algebras(quantum groups).

In the Geer and Patureau's paper, the starting point is a super Lie algebra of type one g. Using the representation theory of the quantum enveloping algebra $U_h(g)$, they obtain a Ribbon Category $\mathcal{M}$. Once with this data, one could think to apply the classical Reshetikhin-Turaev construction. But for this case, the invariants provided by that are often zero. More specifically, there are two kinds of representations in the Ribbon Category $\mathcal{M}$: typical and atypical. In one specific sense, the typical modules form a dense subset in the set of all modules. The problem comes from the fact that the quantum dimension of any typical module vanishes. The Reshetikhin-Turaev construction evaluated on any link with a preferred strand is equal with the quantum dimension of the preferred color times the scalar obtained by evaluating the functor on the tangle obtained from the link by cutting that particular strand. This means that the $\mathcal{M}$ Reshetikhin-Turaev invariants are zero on any link which has at least one strand colored with a typical color. The main idea of this paper is to define another function on the set of typical colors such that applying the same kind of definition but using this function to obtain link invariants. This function will be replacement of the usual quantum dimension and it will be called the "renormalized quantum dimension".

The new invariants initially have values almost in formal power series $\mathbb{C}[[h]]$. In the second part of the paper, the authors construct some multivariable polynomial invariants coming from this renormalized construction. The importance of these polynomial invariants can be seen from the fact that they are strongly related with many previously known polynomial type invariants. Namely, one specialization of the renormalized invariants recovers the multivariable Alexander polynomials. Moreover, they recover the ADO [1] invariants and they are a generalization of the invariants defined by Links and Gould. Also, they have non-trivial intersection with the HOMFLY-PT polynomials and this intersection contains the Kashaev's invariants.

This thesis is organized in 5 Chapters. The first two chapters contain some facts and results concerning the representation theory of a super Lie algebra of type one and the classical Reshetikhin-Turaev construction.

The main part of the thesis consists in the next two chapters. In the third one, we introduce the renormalized construction and in the fourth one we define the multivariable polynomial invariants coming from this.

The last chapter contains some examples concerning the trefoil knot and the Hopf and multi-Hopf links. The thesis finishes with a short appendix and some further directions of study.

Chapter 1

Preliminaries

In this chapter we will introduce some definitions, notations and results which we will use in the sequel.

1.1 Super Lie algebras of type I

A *super-space* is a $\mathbb{C}$-vector space which is $\mathbb{Z}_2$-graded. For a homogeneous element $x \in V = V_0 \oplus V_1$, we denote its parity by $\bar{x} \in \mathbb{Z}_2$ and we call x odd/ even if its parity is 1/0.

Definition 1.1.1 *A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,]\ : g^{\otimes 2} \to g$ which satisfies:*
1) $[\ x, y]\ = -(-1)^{\bar{x}\bar{y}}[\ y, x]$
2) Super Jacobi Identity: $[\ x, [\ y, z]\ ]\ = [\ [\ x, y]\ , z]\ + (-1)^{\bar{x}\bar{y}}[\ y, [\ x, z]\ ]$

Denote by b the distinguished Borel sub-superalgebra of g such that $b = \mathfrak{H} \oplus n_+$, where $\mathfrak{H}$ is a Cartan sub-superalgebra and $g = n_- \oplus \mathfrak{H} \oplus n_+$. Let Δ the set of roots, $\Delta^{\pm}$ be the set of positive/negative roots and Δ_0, Δ_1 the sets of even respective odd roots. A positive root is called simple if it cannot be written as a sum of two positive roots. Also, consider the Weyl group W of g_0.

We will denote by:

$$\rho_0 = \frac{1}{2} \sum\nolimits_{\lambda \in \Delta_0^+} \lambda$$

$$\rho_1 = \frac{1}{2} \sum\nolimits_{\lambda \in \Delta_1^+} \lambda$$

$$\rho = \rho_0 - \rho_1$$

Denote by (A, s) the Cartan matrix coming from the super Lie algebra g and the distinguished Borel sub-super algebra b

Proposition 1.1.2 *a) There exist* $d_1, ..., d_r \in \{\pm 1, \pm 2\}$ *such that the matrix* $(\ d_i a_{ij})\ _{i,j\in\{1...r\}}$ *is symmetric. Moreover, we can assume that* $d_1 = 1$.
b) In a convenient base, the Killing form on $\mathfrak{H}$ *is determined by:*

$$< h_i, h_j >= d_j^{-1} a_{ij}$$

With the notations above, we have the following description of the super Lie algebra g (from [11])

Proposition 1.1.3 *There are elements* $e_i \in n_+$, $f_i \in n_-$ *and* $h_i \in \mathfrak{H}$ *such that* g *is generated by* e_i,f_i *and* h_i *with the relations:*

$$[\ e_i, f_j]\ = \delta_{ij} h_i \qquad [\ h_i, h_j]\ = 0$$

$$[\ h_i, e_j]\ = a_{ij} e_j \qquad [\ h_i, f_j]\ = -a_{ij} f_j.$$

There are two families of super Lie algebras of type I: $sl(m, n)$ and $osp(2, 2n)$. We will always assume that $m \neq n$ and $r = m + n - 1$ in the first situation and $r = n + 1$ in the second one.

For $sl(m, n)$- the Cartan matrix has the following non-zero elements:

$$a_{i,i} = 2 \ except\ a_{m,m} = 0$$

$$a_{i,i+1} = -1 \ except\ a_{m,m+1} = 1$$

$$a_{i+1,i} = -1,$$

and

$$d_i = 1 \ for\ i \in \{1, ..., m\}$$

$$d_i = -1 \ for\ i > m.$$

For $osp(2|2n)$- the Cartan matrix has the following non-zero elements:

$$a_{i,i} = 2 \ except\ a_{1,1} = 0$$

$$a_{i,i+1} = -1 \ except\ a_{1,2} = 1 \ and\ a_{n,n+1} = -2$$

$$a_{i+1,i} = -1,$$

and

$$d_1 = 1$$

$$d_i = -1 \ for\ i \in \{2, ..., n\}$$

$$d_{n+1} = -2.$$

1.2 Representation theory

Let g be a super Lie algebra of type I and $V = V_0 \oplus V_1$ a $\mathbb{Z}_2$-graded vector space.

Notation 1.2.1 *$End_0(V) = \{f \in End(V) \mid f$ preserves the $\mathbb{Z}_2 - grading\}$*
$End_1(V) = \{f \in End(V) \mid f$ switches the $\mathbb{Z}_2 - grading\}$

Definition 1.2.2 *A representation of g on V is a Lie algebras morphism $\pi : g \to End(V)$ such that: $\pi(g_0) \subseteq End_0(V)$ and $\pi(g_1) \subseteq End_1(V)$.*

For $\lambda \in \mathfrak{H}^*$ there is an irreducible module denoted by $V(\lambda)$ such that there exists $v_0 \in V(\lambda)$ with the following properties:
- $hv_0 = \lambda(h)v_0,\ \forall h \in \mathfrak{H}$
- $n_+v_0 = 0$.
Such a module $V(\lambda)$ is called *highest weight module of weight* λ , and v_0 is a *highest weight vector* . Let us denote: $a_i = \lambda(h_i)$.

Proposition 1.2.3 *A highest weight module $V(\lambda)$ is finite dimensional if and only if $a_i \in \mathbb{N}$ for $\forall i \in \{1, ..., r\}, i \neq s$.*

In the sequel a very important role will be played by the so-called *typical modules* :

Definition 1.2.4 *A typical module is a highest weight module $V(\lambda)$ which satisfies:*

$$< \lambda + \rho, \alpha > \neq 0,$$

for all $\alpha \in \triangle_1^+$.
In this case λ is called typical.

Remark 1.2.5 *Let $a_i \in \mathbb{N}$ for $\forall i \in \{1, ..., r\}, i \neq s$. If a weight λ with $\lambda(h_i) = a_i\ \forall i \in \{1, ..., r\}, i \neq s$ is atypical then $\lambda(a_s)$ is in a finite set $\subseteq \mathbb{C}$. Moreover, if λ is atypical then $\lambda(h_s) \in \mathbb{Z}$.*

Notation 1.2.6 *Let $\Lambda = \mathbb{Z}^{r-1} \times \mathbb{C}$ the set of weights taking integer values on h_i for $i \neq s$.*

1.3 Character theory

In this paragraph, we will describe some facts concerning the character and super-character theory of typical g-modules. Let $A = \{f : \mathfrak{H}^* \to \mathbb{C}\}$. We have an application $\mathfrak{H}^* \to A$ sending λ to e^λ, where e^λ is the characteristic function of λ. The above map can be extended to an embedding

$$\mathbb{Z}[\ \Lambda] \ \to A.$$

Remark 1.3.1 *For a weight $\lambda \epsilon \mathfrak{H}^*$, $\frac{1}{2}\lambda \epsilon \mathfrak{H}^*$ so $e^{\frac{1}{2}\lambda}$ is an element of A*

Let us define the following elements of A:

$$L'_0 = \prod_{\alpha \in \Delta_0^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$$

$$L_1 = \prod_{\alpha \in \Delta_1^+} (e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}})$$

$$L'_1 = \prod_{\alpha \in \Delta_1^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})$$

Remark 1.3.2 *The elements L'_0, L_1 and $L'_1 \in \mathbb{Z}\ [\ \Lambda]$*

Definition 1.3.3 *Let V be an irreducible highest weight g-module and*

$$V = \sum\nolimits_\lambda V_\lambda$$

the decomposition with respect to $\mathfrak{H}$, where

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v, \forall h \in \mathfrak{H}\}.$$

The character and super-character of V are:

$$ch(V) = \sum\nolimits_\lambda dim(V_\lambda)e^\lambda$$

$$sch(V) = \sum\nolimits_\lambda (-1)^{deg\lambda} dim(V_\lambda)e^\lambda.$$

For $V(\lambda)$ a typical g-module, we have the the following result from [11]:

Proposition 1.3.4

$$ch(V(\lambda)) = \frac{L_1}{L_0'} \sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)}$$

$$sch(V(\lambda)) = \frac{L_1'}{L_0'} \sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho)}$$

where $\epsilon : W \to \{-1, 1\}$, $\epsilon(w) = \sharp$ *reflections which appear in the word* w.

Using the above formulas, we have the following description for ch and sch from [6]:

Proposition 1.3.5 *For a typical* g*-module* $V(\lambda)$, $ch(V(\lambda)) = \chi_1\chi_0(\lambda)$ *and* $sch(V(\lambda)) = \chi_1'\chi_0(\lambda)$, *where*

$$\chi_1' = \prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha}), \ \chi_1 = \prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha}).$$

1.4 The quantum enveloping algebra

In this part, we will describe the quantum deformation of a super Lie algebra of type I and some of its properties. First of all, let us introduce some notations: set h be an indeterminate, $q = e^{\frac{h}{2}}$ and $z \ = q^z - q^{-z}$.

Definition 1.4.1 *The quantization of a super Lie algebra* g *of type I, denoted by* $U_h(g)$ *is the* $\mathbb{C}[\ [h]\ ]$ *-super-algebra generated by three families of elements* h_i, E_i *and* F_i, *for* $i \in \{1, ..., r\}$ *with the relations:*

$$[\ h_i, h_j] \ = 0 \qquad [\ E_i, F_j] \ = \delta ij \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$$

$$[\ h_i, E_j] \ = a_{ij} E_j \qquad [\ h_i, F_j] \ = -a_{ij} F_j \qquad {E_s}^2 = {F_s}^2 = 0$$

and quantum Serre type relations, where $[\ x, y] \ = xy - (-1)^{\bar{x}\bar{y}} yx$.

Proposition 1.4.2 $U_h(g)$ *is a Hopf super-algebra with the following coproduct, counit and antipode:*

$$\Delta(E_i) = E_i \otimes 1 + q^{-h_i} \otimes E_i \qquad \epsilon(E_i) = 0 \qquad S(E_i) = -q^{h_i} E_i$$

$$\Delta(F_i) = F_i \otimes q^{h_i} + 1 \otimes F_i \qquad \epsilon(F_i) = 0 \qquad S(F_i) = -F_i q^{-h_i}$$

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \qquad \epsilon(h_i) = 0 \qquad S(h_i) = -h_i$$

.

An important fact proved in [14] and [21] is that $U_h(g)$ has an R-matrix explained bellow:

- consider one normal ordering on the set of roots Δ^+
- let E_α, F_α the Cartan-Weyl generators for the quantization $U_h(g)$ such that $E_{\alpha_i} = E_i$ and $F_{\alpha_i} = F_i$ for the simple roots
- let $\{e_\alpha, f_\alpha, h_i\}$ the Cartan-Weyl basis of g
- denote by $h_\alpha = [\ e_\alpha, f_\alpha]\ \in g$ where $\alpha \in \Delta^+$
- let a_α defined by:

$$[\ E_\alpha, F_\alpha]\ = a_\alpha \frac{(q^{h_\alpha} - q^{-h_\alpha})}{q - q^{-1}}$$

- denote by $\check{R}_\alpha = exp_q((-1)^{\bar{\alpha}} {a_\alpha}^{-1}(q - q^{-1})(E_\alpha \otimes F_\alpha)) \in U_h(g) \otimes U_h(g)$
-

$$\check{R} = \prod_{\alpha \in \Delta^+} \check{R}_\alpha \in U_h(g) \otimes U_h(g)$$

- consider the matrix (d_{ij}) the inverse of $(a_{ij}{d_j}^{-1})$
- let

$$K = q^{\sum_{i,j} d_{ij} h_i \otimes h_j} \in U_h(g) \otimes U_h(g)$$

With these notations, the result of Khoroshkin, Tolstoy and Yamane is:

Theorem 1.4.3 *For any super Lie algebra of type I, the quantum Hopf super-algebra $U_h(g)$ has an R-matrix which verifies:*

$$R = \check{R}K.$$

In what follows, we will consider a particular class of $U_h(g)$-modules which will be used in the construction of the link invariants.

Definition 1.4.4 *An $U_h(g)$-module W is called topologically free of finite rank if there is a finite dimensional g-module V such that $W \simeq V[\ [\ h]\ ]$ as $\mathbb{C}[\ [\ h]\ ]$ -modules.*

Let $\mathcal{M}$ the category of topologically free of finite rank $U_h(g)$-modules. An important fact from [5] is the following:

Proposition 1.4.5 *$\mathcal{M}$ is a Ribbon Category.*
It has a brading $C = \tau \circ R$, where $\tau(v \otimes w) = (-1)^{\bar{v}\bar{w}} w \otimes v$.
The duality morphisms for $V \in \mathcal{M}$ are defined as follows:
$b_V : \mathbb{C}[\ [\ h]\ ]\ \to V \otimes V^*$ *and* $d'_V : V \otimes V^* \to \mathbb{C}[\ [\ h]\ ]$
$b_V(1) = \sum_i v_i \otimes {v_i}^*$ *for* $\{v_i \mid i \in 1, ..., n\}$ *a basis of* V
$d'_V(v \otimes f) = (-1)^{\bar{v}\bar{f}} f(q^{2<\eta,\rho>}v)$, *where v is a vector of weight η.*

Chapter 2

The Reshetikhin-Turaev construction

In 1991, Reshetikhin and Turaev defined a construction which starts with any Ribbon Category $\mathcal{C}$ and obtains framed $\mathcal{C}$-colored links tangles invariants. More precisely, they defined a monoidal functor from the category of C-colored framed tangles to C. The main idea is to prescribe the value of the functor on elementary tangles. Once this is known, starting with any tangle, by splitting this in "small pieces" and using the functoriality and monoidality of the construction, one can deduce the value of the functor on it. This is an important method which is often applied for a category which comes from the representation theory of a quantum group.
In the sequel, we will define the notions that appear in this construction and we will give the precise statement of the theorem.

2.1 Ribbon categories

Definition 2.1.1 *Let $\mathcal{C}$ a strict monoidal category.*
1)A braiding C is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \rightarrow W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:

$$C_{U,V\otimes W} = (Id_V \otimes C_{U,W}) \circ (C_{U,V} \otimes Id_W)$$
$$C_{U\otimes V,W} = (C_{U,W} \otimes Id_V) \circ (Id_U \otimes C_{V,w}).$$

2)If $\mathcal{C}$ has the brading C, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$:

$$\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W).$$

3)We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$, $d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties:

$$(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$$

$$(d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}.$$

4)The duality is said to be compatible with the brading and the twist if:

$$(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V, \ \forall V \in \mathcal{C}.$$

5) A category with a brading, a twist and a compatible duality is called a Ribbon Category.

2.2 Category of tangles

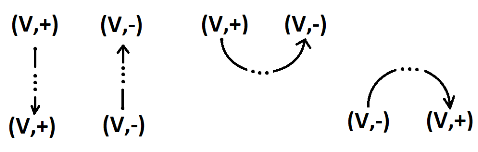
Definition 2.2.1 *Consider $\mathcal{C}$ be a strict category.*
The category of $\mathcal{C}$-colored framed tangles $\mathcal{T}_{\mathcal{C}}$ is defined as follows:

$$Ob(\mathcal{T}_{\mathcal{C}}) = \{(V_1, \epsilon_1), ..., (V_m, \epsilon_m) \mid m \in \mathbb{N}, \epsilon_i \in \{\pm 1\}, V_i \in \mathcal{C}\}.$$

$$Morph(\mathcal{T}_{\mathcal{C}})((V_1, \epsilon_1), ..., (V_m, \epsilon_m), (W_1, \delta_1), ..., (W_n, \delta_n)) =$$

$$= \frac{\mathcal{C} - colored\ framed\ tangles\ T : (V_1, \epsilon_1), ..., (V_m, \epsilon_m) \to (W_1, \delta_1), ..., (W_n, \delta_n)}{isotopy}.$$

Observation : The tangles have to respect the colors V_i. Once we have such a tangle, it has an induced orientation, coming from the signs ϵ_i, using the following conventions:



The composition of morphisms in this category corresponds to the superposition of the tangles. The tensor product of two objects means to construct the union of the representing sequences and the tensor product of morphisms means to juxtapose the corresponding tangles. Also, there is an unitary object: $(\mathbf{1}, +) \in \mathbf{Ob}(\mathcal{T}_{\mathcal{C}})$ and for any $V \in \mathcal{C}$, the identity morphism of $(V, +)$ is represented by the trivial tangle going downwards from V to V. Moreover any object $(V, +)$ has the dual $(V, -)$.
This shows that $\mathcal{T}_{\mathcal{C}}$ is a strict monoidal category.

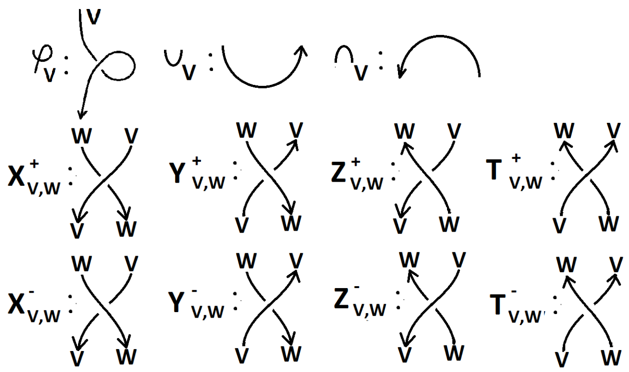
Proposition 2.2.2 *The category of tangles $\mathcal{T}_{\mathcal{C}}$ admits a brading, a twist and a compatible duality which means that this is a Ribbon Category.*

2.3 The Reshetikhin-Turaev functor

Theorem *(Reshetikhin-Turaev) Consider($\mathcal{C}$, C, Θ, b, d') a Ribbon category. Then there exist an unique functor $F_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$ which is monoidal and satisfies the following local relations for any $V, W \in \mathcal{C}$:*
1) $F((V, +)) = V \quad F((V, -)) = (V)^*$
2)

$$F(\varphi_V) = \theta_V \quad F(\cup_V) = b_V \quad F(\cap_V) = d'_V$$
$$F(X^+_{V,W}) = C_{V,W} \qquad F(X^-_{V,W}) = (C_{V,W})^{-1}$$
$$F(Y^+_{V,W}) = (C_{W,V})^{-1} \qquad F(Y^-_{V,W}) = C_{V^*,W}$$
$$F(Z^+_{V,W}) = (C_{(V)^*,W})^{-1} \qquad F(Z^-_{V,W}) = C_{V,W^*}$$
$$F(T^+_{V,W}) = C_{(V)^*,W^*} \qquad F(T^-_{V,W}) = (C_{(W)^*,(W)^*})^{-1}$$

where



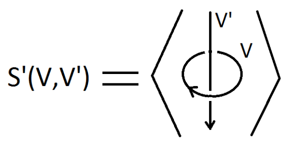
Chapter 3

The modified Reshetikhin-Turaev type construction

This chapter is devoted to the Geer-Patureau construction of the non-trivial link invariants using the renormalized quantum dimension for super-Lie algebras of type I.

3.1 The renormalized quantum dimension

Definition 3.1.1 *Let F be the Reshetikhin-Turaev functor for the Ribbon category $\mathcal{M}$.*
1)For $V \in \mathcal{M}$ irreducible and $T \in \mathcal{T}(V,V)$: $F(T) = x \cdot Id$, $F(T) \in End_{U_h(g)}(V)$. We denote by $< T >= x$ and call it the bracket of T.
2)For $V, V' \in \mathcal{M}$ and V' irreducible, we define:



3) For $V = \tilde{V}(\lambda)$ and $V' = \tilde{V}(\mu)$ two irreducible highest weight $U_h(g)$-modules which are deformations of $V(\lambda)$ and $V(\mu)$, we denote: $S'(\lambda, \mu) = S'(V, V')$.

We will consider the following function $\varphi_\beta : \mathbb{Z}[\Lambda] \to \mathbb{C}[[h]]$ defined by the expression

$$\varphi_\beta(e^\alpha) = e^{<\alpha,\beta>h},$$

for a fixed weight $\beta \in \mathfrak{H}^*$. Consider $\mathbb{Z}[[\Lambda]]$ as a ring with the multiplication given by the convolution operation, which can be defined on the generators and after extended linearly: for $\alpha,\gamma \in \Lambda$,

$$e^\alpha \star e^\gamma = e^{\alpha+\gamma}.$$

If we look at this ring structure on $\mathbb{Z}[[\Lambda]]$, φ_β becomes a morphism. The next proposition gives an expression of S' using the function φ and the super-character:

Proposition 3.1.2

$$S'(\lambda,\mu) = \varphi_{\mu+\rho}(sch(V(\lambda))).$$

Proof: Consider $\{v_i\}$ a basis for $V\tilde{(\lambda)}$ with v_i a weight vector of weight $\alpha_i \in \mathfrak{H}^*$ and $w_\mu \in \tilde{V}(\mu)$ a highest weight vector (of weight μ). A first observation is that we have the following facts:

Fact 0:

$$K(w_\mu \otimes v_i) = q^{<\mu,\alpha_i>} w_\mu \otimes v_i \qquad K(v_i \otimes w_\mu) = q^{<\mu,\alpha_i>} v_i \otimes w_\mu$$

Fact 1:

$$R(w_\mu \otimes v) = q^{<\mu,\eta>}(w_\mu \otimes v),$$

which comes from the formula of the R-matrix, from Fact 0 and from the property of w_μ to be a highest weight vector, more specifically $E_\alpha w_\mu = 0$ for all $\alpha \in \Delta^+$.

Fact 2:

$$\check{R}(v \otimes v_\mu) = v \otimes v_\mu + \sum v' \otimes w'$$

with $v' \in \tilde{V}(\lambda)$, $w' \in \tilde{V}(\mu)$ are weight vectors and the weight of v' has a strictly higher order than the weight of v (which means that the difference is a positive root). This follows from the fact that ${E_\alpha}^n(v)$ is zero or a vector with weight of strictly higher order than v.

Indeed: let $v' = E_\alpha v$. We have $[H, E_\alpha] = \alpha(H)E_\alpha$, so $HE_\alpha = E_\alpha(H + \alpha(H))$. Looking at the action of H on v' we have:

$$Hv' = HE_\alpha v_\lambda = E_\alpha(H + \alpha(H))v_\lambda =$$

$$E_\alpha(\lambda(H)v_\lambda + \alpha(H)v_\lambda) = (\lambda(H) + \alpha(H))v' = (\lambda + \alpha)(H)v'.$$

So, v' has the weight $\lambda + \alpha$.

In the second part we will use the structural morphisms of the Ribbon category $\mathcal{M}$ in order to compute $S'(\lambda, \mu)$. More specifically, it has: morphisms for duality $b_V : \mathbb{C}[\,[h]\,] \to V \otimes V^*$, $d'_V : V \otimes V^* \to \mathbb{C}[\,[h]\,]$ and the braiding $c_{V,V'} : V \otimes V' \to V' \otimes V$. They are defined as follows:

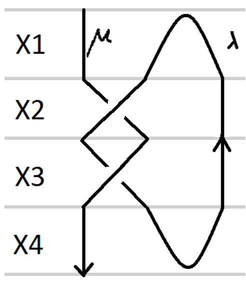
$$b_{\tilde{V}(\lambda)}(1) = \sum v_i \otimes {v_i}^*$$

$$d'_{\tilde{V}(\lambda)}(v \otimes f) = (-1)^{\bar{v}\bar{f}} f(q^{2<\eta,\rho>}v),$$

where v is a vector of weight η and

$$c_{V,W} = \tau \circ R,$$

where τ is the twist. Let the following endomorphism $S \in End_{U_h(g)}(V\tilde{(}\mu))$, the image through the Reshetikhin-Turaev functor of the following tangle:



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We have $S = X_1 \circ X_2 \circ X_3 \circ X_4$, where

$$X_1 = Id_{\tilde{V}(\mu)} \otimes d'_{\tilde{V}(\lambda)} \qquad X_2 = c_{\tilde{V}(\lambda),\tilde{V}(\mu)} \otimes Id_{\tilde{V}(\lambda)^*}$$

$$X_3 = c_{\tilde{V}(\mu),\tilde{V}(\lambda)} \otimes Id_{\tilde{V}(\lambda)^*} \qquad X_4 = Id_{\tilde{V}(\mu)} \otimes b_{\tilde{V}(\lambda)}$$

In order to compute $S'(\lambda, \mu)$, it is sufficient to find out the value of S on any particular vector. For us, it is convenient to look at w_μ. We have the following relations:

$$S(w_\mu) = S'(\lambda, \mu)w_\mu$$

$$S(w_\mu) = X1 \circ X2 \circ X3(w_\mu \otimes \sum_i (v_i \otimes {v_i}^*)) =$$

$$= X1 \circ X2(\sum_i q^{<\mu,\alpha_i>} v_i \otimes w_\mu \otimes {v_i}^*) =$$

$$=^* X1(\sum_i (q^{2<\mu,\alpha_i>} w_\mu \otimes v_i \otimes {v_i}^*) + \sum_i (\sum_k ({w_k}' \otimes {v_k}' \otimes z_k))) =$$
$$= \sum_i (-1)^{\bar{v_i}} q^{2<\mu+\rho,\alpha_i>} w_\mu,$$

$*$ For the third equality, we applied Fact 2: ${v_k}' \in \tilde{V}(\lambda)$ is a weight vector of weight strictly higher than the weight of v_i.
So, we obtain that:

$$S'(\lambda,\mu) = \sum_i (-1)^{\bar{v_i}} q^{2<\mu+\rho,\alpha_i>}.$$

On the other hand, we have:

$$\varphi_{\mu+\rho}(sch(V(\lambda))) = \varphi_{\mu+\rho}(\sum_i (-1)^{\bar{\alpha_i}} e^{\alpha_i}) = \sum_i (-1)^{\bar{\alpha_i}} q^{2<\alpha_i,\mu+\rho>}.$$

From the two previous relations, we have proved the Proposition. □

Corollary 3.1.3 *For a typical weight λ: $S'(\lambda,\mu) = 0$ if and only if μ is an atypical weight.*

Proof: Using *Proposition* 3.1.2, *Proposition* 1.3.5 and the fact that $\varphi_.$ is multiplicative we have:

$$S'(\lambda,\mu) = \varphi_{\mu+\rho}(sch(V(\lambda))) = \varphi_{\mu+\rho}(\prod_{\alpha\in\Delta_1^+}(1-e^{-\alpha})\chi_0(\lambda)) =$$

$$= \prod_{\alpha\in\Delta_1^+}(1-\varphi_{\mu+\rho}(e^{-\alpha}))\varphi_{\mu+\rho}(\chi_0(\lambda))$$

But, using the definition $\chi_0(\lambda) = \sum_\mu dim(V(\mu))\ e^\mu$ where $dimV(\mu) > 0$, we obtain that

$$\varphi_{\mu+\rho}(\chi_0(\lambda)) > 0.$$

So $S'(\lambda,\mu) = 0 \Leftrightarrow \exists\alpha \in {\Delta_1}^+$ such that $\varphi_{\mu+\rho}(e^{-\alpha}) = 1$,which means that

$$< \mu+\rho, \alpha >= 0.$$

Using $Theorem 1$ from [11] one can obtain that:
μ is atypical $\Leftrightarrow \exists\alpha \in {\Delta_1}^+$ such that $< \mu+\rho,\alpha >= 0$.
From the last two paragraphs, the conclusion holds. □

The next Lemma, is the fundamental fact which helps us to define the modified quantum dimension:

Lemma 3.1.4 *Consider* λ *a typical weight. Then the expression*

$$\frac{\varphi_{\lambda+\rho}(L_0')}{\varphi_{\lambda+\rho}(L_1')\varphi_\rho(L_0')}$$

is in $h^{-k_g}\mathbb{C}[\,[\,h]\,]$ *, where* k_g *is the number of odd positive roots.*

As a consequence of the previous Lemma, one can introduce the following:

Definition 3.1.5 *Let* $d : \{\lambda \mid \lambda \in \mathfrak{H}^* \text{ typical weight}\} \to \mathbb{C}[\,[\,h, h^{-1}]\,]$ *defined by*

$$d(\lambda) = \frac{\varphi_{\lambda+\rho}(L_0')}{\varphi_{\lambda+\rho}(L_1')\varphi_\rho(L_0')}.$$

The function d is called the modified quantum dimension and will be used as a replacement of the usual quantum dimension of a module in order to construct the modified link invariants.

3.2 The modified link invariants

We begin this section with a relation between the modified quantum dimension and the S' function.

Lemma 3.2.1 *For two typical weights* λ *and* μ*, we have:*

$$d(\mu)S'(\lambda,\mu) = d(\lambda)S'(\mu,\lambda).$$

Proof: Using *Proposition* 3.1.2 and *Theorem*1 from [11] which gives one formula for the super-character using the Weyl group, we have:

$$d(\mu)\ S'(\lambda,\mu) = d(\mu)\varphi_{\mu+\rho}(sch(V(\lambda))) =$$

$$= \frac{\varphi_{\mu+\rho}(L_0')}{\varphi_{\mu+\rho}(L_1')\varphi_\rho(L_0')} \cdot \varphi_{\mu+\rho}\Big(\frac{L_1'}{L_0'}\sum\nolimits_{w\in W}\epsilon(w)\ e^{w(\lambda+\rho)}\Big) =$$

$$= \frac{1}{\varphi_\rho(L_0')}\ \sum\nolimits_{w\in W}\epsilon(w)\ e^{<w(\lambda+\rho),\mu+\rho>} =$$

Analog, we have that

$$d(\lambda)\ S'(\mu,\lambda) = \frac{1}{\varphi_\rho(L_0')}\ \sum\nolimits_{w\in W}\epsilon(w)\ e^{<w(\mu+\rho),\lambda+\rho>}$$

But the scalar product $< , >$ is invariant at the action of W ($< w(a), b >=< a, w(b) >$), so the following equality holds:

$$< w(\lambda + \rho), \mu + \rho >=< w(\mu + \rho), \lambda + \rho > .$$

Using the last three relations, we obtain the conclusion. $\square$

Before stating the following result, we need some classical notations and results which will be useful:

Definition 3.2.2 *1)Let $m, n \in \mathbb{N}$. A partition λ of length m for n is a finite sequence: $\lambda = (\lambda_1, ..., \lambda_m)$ such that $n \geq \lambda_1 \geq ... \geq \lambda_m \geq 0$*
2)If $n \in \mathbb{N}$ and λ a partition with $\lambda_1 \leq n$, the complementary partition is defined by: $\hat{\lambda} = (\hat{\lambda_1}, ..., \hat{\lambda_m})$, where $\hat{\lambda_i} = n - \lambda_{m+1-i}$.
3)For a partition λ, the conjugate partition is defined as: $\hat{\lambda}' = (\hat{\lambda'_1}, ..., \hat{\lambda'_m})$ where $\hat{\lambda'_i} = card\{j | j \in 1, .., m; \lambda_j \geq i\}$.
4)Let λ a partition with length m. The Schur polynomial associated to λ is:

$$s_\lambda(x) := s_\lambda(x_1, ..., x_m) = \frac{det(x_j^{\lambda_i+m-i})_{i,j}}{det(x_j^{m-i})_{i,j}}$$

Proposition 3.2.3 *Cauchy Identity:*
Let $m, n \in \mathbb{N}$. The following relation holds:

$$\prod_{i \in 1,...,m, j \in 1,...,n} (x_i + y_j) = \sum\nolimits_{\lambda \subset n^m} s_\lambda(x) s_{\hat{\lambda}'}(y)$$

where the sum is over all partitions for n of length m.

Lemma 3.2.4 *There is a typical weight $\lambda_0 \in \mathfrak{H}^*$ such that $End_{U_h(g)}(\tilde{V}(\lambda_0) \otimes \tilde{V}(\lambda_0))$ is a commutative ring.*

Proof: We will prove that there is a weight λ_0 such that $V(\lambda_0) \otimes V(\lambda_0)$ splits as a direct sum of distinct irreducible modules. This will be done with character theory, namely we will write the character of the tensor product and we will express this as a sum of characters of distinct modules.
Consider $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Define the weight $\lambda_0 \in \mathfrak{H}^*$ determined by: $\lambda_0(h_i) = 0$ for any $i \neq s$ and $\lambda_0(h_s) = \alpha$.
From $Proposition 1.3.5$, we have that:

$$ch(V(\lambda_0)) = \chi_1 \chi_0(\lambda_0),$$

where $\chi_1 = \prod_{\alpha\in\Delta_1^+} (1+e^{-\alpha})$ and $\chi_0(\lambda_0)$ is the character of the irreducible g_0-module $V_0(\lambda_0)$. From the definition of λ_0, we have: $dim(V_0(\lambda_0)) = 1$ so $\chi_0(\lambda_0) = e^{\lambda_0}$. Using the formula which relates characters and tensor products and the previous relation, we get:

$$ch(V(\lambda_0) \otimes V(\lambda_0)) = ch(V(\lambda_0))ch(V(\lambda_0)) = \chi_1 \cdot (\chi_1 e^{2\lambda_0}).(1)$$

On the other hand, if $V(\lambda_0) \otimes V(\lambda_0) = \oplus V(\lambda_i)$ then:

$$ch(V(\lambda_0) \otimes V(\lambda_0)) = \sum ch(V(\lambda_i)) = \chi_1 \sum \chi_0(\lambda_i).(2)$$

From the previous two relations $(1), (2)$, we aim to express $\chi_1 e^{2\lambda_0}$ as a sum of characters of distinct irreducible g_0-modules.

Let us consider the $sl(m|n)$ case.
We have: $\mathfrak{H}^* =< \epsilon_i, \delta_j | i \in 1, ..., m; j \in 1, ..., n > /str$, where $str = \sum_i \epsilon_i - \sum_j \delta_j$. The odd positive roots are expressed as follows:

$$\Delta_1^+ = \{\epsilon_i - \delta_j |\ 1 \leq i < j \leq m\}.$$

Denote by: $x_i = e^{\epsilon_i}$ and $y_j = e^{\delta_j}$. Then:

$$\chi_1 = \prod_{\alpha\in\Delta_1^+} (1+e^{-\alpha}) = \prod_{i,j} (1+\frac{y_j}{x_i}) = (x_1...x_m)^{-n} \prod_{i,j} (x_i + y_j).$$

Using the Cauchy's Identity, we get:

$$\chi_1 = (x_1...x_m)^{-n} \sum\nolimits_{\lambda\subset n^m} s_\lambda(x) s_{\hat{\lambda}'}(y)$$

$$\Rightarrow \chi_1\ e^{2\lambda_0} = e^{2\lambda_0}(x_1...x_m)^{-n} \sum\nolimits_{\lambda\subset n^m} s_\lambda(x) s_{\hat{\lambda}'}(y)(3)$$

There is the following known result: $g_0 \simeq sl(m) \times sl(n) \times \mathbb{C}$.

Also, from [3] *Prop* 15.15, we have the following result:

Lemma *For any partition $\lambda = (\lambda_1, ..., \lambda_m)$, there exist an irreducible representation S_λ of $sl(m)$ such that the function on $\mathfrak{H}^*$: $s_\lambda(x)$ is the character of S_λ. Moreover, two $sl(m)$ -representation S_λ and S_μ are isomorphic if and only if $\lambda_1 - \mu_1 = ... = \lambda_m - \mu_m$.*

From the structure of g_0, if we have $V_m(\lambda_m)$ an irreducible $sl(m)$-module of highest weight λ_m, $V_n(\lambda_n)$ an irreducible $sl(n)$-module of highest weight λ_n and W is an irreducible $\mathbb{C}$-module then:
$V = V_m(\lambda_m) \otimes V_n(\lambda_n) \otimes W$ is an irreducible g_0 representation. (4)

Using relations $(3), (4)$ and the **Lemma** , we obtain that:

$$\Rightarrow \chi_1 \ e^{2\lambda_0} = ch(W) \sum\nolimits_{\lambda \subset n^m} s_\lambda(x) s_{\hat{\lambda}'}(y) = \sum\nolimits_{\lambda \subset n^m} \chi_0(S_\lambda \otimes S_{\hat{\lambda}'} \otimes W).$$

In order to conclude that all modules that appear in the sum above are different, we will investigate their $sl(m)$ and $sl(n)$ components.
We have the following decomposition:

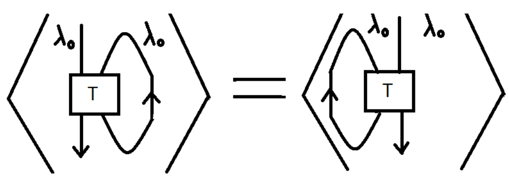
$$\chi_1 \ e^{2\lambda_0} = \sum_{i=0}^{mn} \sum\nolimits_{\lambda \subset n^m |\lambda|=i} \ \chi_0(S_\lambda \otimes S_{\hat{\lambda}'} \otimes W).$$

For any $i \in \{0, ..., mn\}$ we have that for any two distinct partitions λ_1, λ_2 such that $\mid \lambda_1 \mid = \mid \lambda_2 \mid = i$, the corresponding $sl(m)$-modules S_{λ_1} and S_{λ_2} are different. Indeed if we suppose that $S_{\lambda_1} \simeq S_{\lambda_2}$, then there exist $d \in \mathbb{N}$ such that $\lambda_{1,j} - \lambda_{2,j} = d$, for all $j \in 1, ..., m$, so $\mid \lambda_1 \mid = \mid \lambda_2 \mid + md$ which is a contradiction.
Also, for two different values $i_1, i_2 \in \mathbb{N}$ and any partitions λ_1, λ_2 such that $\mid \lambda_1 \mid = \mid \lambda_2 \mid$, the corresponding modules $S(\lambda_1) \otimes S(\hat{\lambda_1'})$ and $S(\lambda_2) \otimes S(\hat{\lambda_2'})$ are different.
As a conclusion, all the g-modules that appear in the previous sum are different, so $V(\lambda_0) \otimes V(\lambda_0)$ splits as a sum of distinct irreducible g_0-modules.

$\square$

Using the deformation of this special object $\tilde{V}(\lambda_0) \in M$, one can deduce the following property:

Lemma 3.2.5 *For any colored tangle* $T \in \mathcal{T}((\tilde{V}(\lambda_0), \tilde{V}(\lambda_0)), (\tilde{V}(\lambda_0), \tilde{V}(\lambda_0)))$ *there is the relation:*



Proof: Denote by $\tilde{T}$ the tangle from the left hand side and $\tilde{\tilde{T}}$ the one from the right hand side. We have:

$$< \tilde{\tilde{T}} > \cdot Id_{\tilde{V}_{\lambda_0}} = F(\tilde{\tilde{T}}) = F(T_1 \circ T_2 \circ T_3) = F(T_1) \circ F(T_2) \circ F(T_3) =$$

$$= (d_{\tilde{V}_{\lambda_0}} \otimes Id_{\tilde{V}_{\lambda_0}}) \circ (Id_{\tilde{V}^*_{\lambda_0}} \otimes F(T)) \circ (b'_{\tilde{V}_{\lambda_0}} \otimes Id_{\tilde{V}_{\lambda_0}})$$

This motivates the following notations: For $f \in End_{\tilde{V}_{\lambda_0}}$ denote by:

$$ptr_L(f) := (d_{\tilde{V}_{\lambda_0}} \otimes Id_{\tilde{V}_{\lambda_0}}) \circ (Id_{\tilde{V}^*_{\lambda_0}} \otimes f) \circ (b'_{\tilde{V}_{\lambda_0}} \otimes Id_{\tilde{V}_{\lambda_0}}) \ \in End(\tilde{V}_{\lambda_0})$$

$$ptr_R(f) := (Id_{\tilde{V}_{\lambda_0}} \otimes d'_{\tilde{V}_{\lambda_0}}) \circ (f \otimes Id_{\tilde{V}^*_{\lambda_0}}) \circ (Id_{\tilde{V}_{\lambda_0}} \otimes b_{\tilde{V}_{\lambda_0}}) \ \in End(\tilde{V}_{\lambda_0})$$

So, we obtain the following relations, denoted by (1):

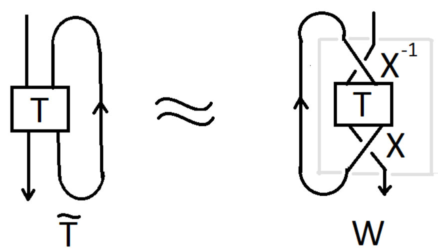
$$< \tilde{\tilde{T}} > \cdot Id_{\tilde{V}_{\lambda_0}} = ptr_L(F(T))$$

$$< \tilde{T} > \cdot Id_{\tilde{V}_{\lambda_0}} = ptr_R(F(T))$$

In the sequel, the following result will be used:
Lemma : $\forall T \in \mathcal{T}((\tilde{\mathcal{V}}_{\lambda'}, \tilde{\mathcal{V}}_{\lambda'}), (\tilde{\mathcal{V}}_{\lambda'}, \tilde{\mathcal{V}}_{\lambda'}))$ there is the relation:

$$ptr_R(F(T)) = ptr_L(c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}}{}^{-1} \circ F(T) \circ c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}})$$

Proof : The two tangles bellow are isotopic, so equivalent:



$$\Rightarrow F(\tilde{T}) = F(W).(2)$$

Using the relations (1) and (2) we deduce that:

$$ptr_R(F(T)) = F(\tilde{T}) = F(W) = ptr_L(F(X^{-1} \circ T \circ X)) =$$

$$= ptr_L(F(X^{-1}) \circ F(T) \circ F(X)) = ptr_L({c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}}}^{-1} \circ F(T) \circ c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}}),$$

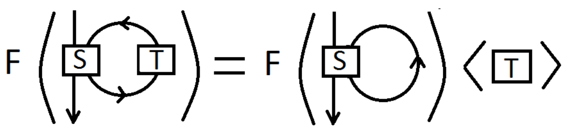
so the relation from the statement is true.

Until now, we did not use the fact that in the hypothesis we have the module $\tilde{V}_{\lambda_0}$ which has the property from *Lemma*3.2.2. Denote by $E := End(\tilde{V}_{\lambda_0} \otimes \tilde{V}_{\lambda_0})$. Then E is a commutative ring. But $F(T), c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}} \in E$ so $F(T) = {c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}}}^{-1} \circ F(T) \circ c_{\tilde{V}_{\lambda_0}, \tilde{V}_{\lambda_0}}$. As a conclusion, using the *Lemma*, we obtain that $ptr_R(F(T)) = pth_L(F(t))$ and using relation (1) we have that $< \tilde{T} >=< \tilde{\tilde{T}} >$ which finish the proof. □

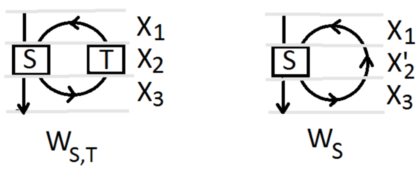
The existence of such a module $V(\lambda_0)$ is important having in mind the previous relation, which plays an essential role in the argument that the Modified Reshetikhin-Turaev construction is invariant with respect to the cutting strand. This idea was generalized in [8] where the authors call an object of a category ambidextrous if it has exactly the property from the Lemma above.

Now, we will state one general property of the Reshetikhin-Turaev functor, which comes from the fact that this is monoidal:

Lemma 3.2.6 *For any two tangles S and T, the following relations holds:*



Proof: We split the two tangles in strips as in the picture:



Then, using the functoriality and the monoidality of the Reshetikhin-Turaev function, we get:

$$F(W_{S,T}) = F(X_1 \circ X_2 \circ X_3) = F(X_1) \circ F(X_2) \circ F(X_3) = F(X_1) \circ (F(S) \otimes F(T)) \circ F(X_3) =$$

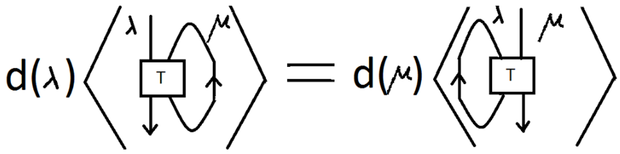
$$F(X_1) \circ (F(S) \otimes Id < T >) \circ F(X_3) =< T > \ F(X_1) \circ (F(S) \otimes Id) \circ F(X_3)$$

$$=< T > \ F(X_1) \circ F(X_2') \circ F(X_3) =< T > F(W_S).$$

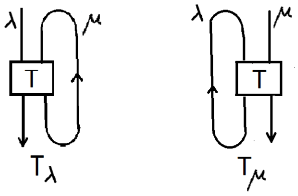
So, the conclusion holds. □

The next Lemma, is the main result towards the construction of the link invariants:

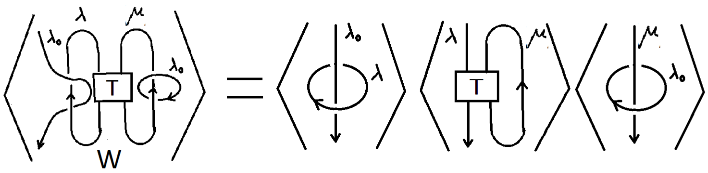
Lemma 3.2.7 *For* $\lambda, \mu \in \mathfrak{H}*$ *typical weights and for any tangle* $T \in \mathcal{T}((\tilde{V}(\lambda), \tilde{V}(\mu)), (\tilde{V}(\lambda), \tilde{V}(\mu)))$ *the following relation holds:*



Proof: Let us make denote by T_λ and T_μ the following links:

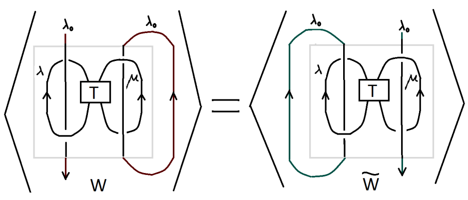


In the proof, we will use one special tangle which has as colors λ,μ and the particular weight λ_0. Using the previous *Lemma*, we have that:



$$= S'(\lambda, \lambda_0) \ < T_\lambda > \ S'(\lambda_0, \mu) \ (1)$$

But applying *Lemma* 3.2.3, we are having:



(2)

By an analogue argument:

$$< \tilde{W} >= S'(\mu, \lambda_0) \ < T_\mu > \ S'(\lambda_0, \lambda) \ (3)$$

From relations $(1), (2)$ and (3), we obtain that:

$$S'(\lambda, \lambda_0) \ S'(\lambda_0, \mu) \ < T_\lambda >= S'(\mu, \lambda_0) \ S'(\lambda_0, \lambda) \ < T_\mu >$$

Multiplying by $d(\lambda)d(\mu)$, we get:

$$(d(\lambda)S'(\lambda, \lambda_0)) \, (d(\mu)S'(\lambda_0, \mu)) \ < T_\lambda >= (d(\mu)S'(\mu, \lambda_0)) \, (d(\lambda)S'(\lambda_0, \lambda)) \ < T_\mu >$$

Using the relation between S' and d from *Lemma* 3.2.1 for (λ_0, μ) in the left hand side and for (λ, λ_0) in the right hand side, we have the following result:

$$(d(\lambda)S'(\lambda, \lambda_0)) \, (d(\lambda_0)S'(\mu, \lambda_0)) \ < T_\lambda >= (d(\mu)S'(\mu, \lambda_0)) \, (d(\lambda_0)S'(\lambda, \lambda_0)) \ < T_\mu >$$

After we simplify the corresponding terms, we obtain:

$$d(\lambda) \ < T_\lambda >= d(\mu) \ < T_\mu >$$

which is the desired conclusion. $\square$

Now, having all previous results in mind, we are able to state the First Main Theorem from the paper [6]:

Theorem 3.2.8 *Let the map F' from the set of framed colored links, colored with at least one typical $U_h(g)$-module with values in $h^{-k_g}\mathbb{C}[\,[\,h]\,]$ defined as follows: for a colored link L with a typical color λ, $L = \widehat{T_\lambda}$ denote*

$$F'(L) = d(\lambda) < T_\lambda > .$$

Then: the map F' is a well defined colored link invariant.

Chapter 4

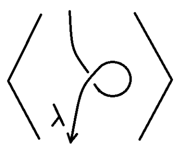
The Geer-Patureau multivariable link invariants

In this chapter, using the previous main theorem, we present the construction of the multi-variable link invariants introduced by Geer and Patureau.

4.1 Technical results

The following result, express the value of the twist $\theta_{\tilde{V}}(\lambda)$

Lemma 4.1.1 *If $\lambda \in h^*$ is a dominant weight. Then:*



$$= q^{<\lambda,\lambda+2\rho>}.$$

In the next Lemmas, we will introduce some notations which will be used in the definition of the multivariable link invariants. Recall that the dominant weights are indexed by $\mathbb{N}^{r-1} \times \mathbb{C}$. We know that for fixed $c \in \mathbb{N}^{r-1}$, the set $\mathbb{T}_c := \{a \mid a \in \mathbb{C}, \lambda_a^c \textit{ is typical}\}$ has a finite complement in $\mathbb{C}$. Denote by λ_a^c the weight corresponding to (c, a) and $\tilde{V}_a^c = \tilde{V}(\lambda_a^c)$.

Lemma 4.1.2 *For a typical weight λ_α^c and for $\alpha \in \Delta^+$, there is $n_\alpha^c \in \mathbb{Z}$ such that:*
1) If we denote by:

$$M_0^c(q) = \prod_{\alpha \in \Delta_0^+} \frac{q^{n_\alpha^c} - q^{-n_\alpha^c}}{q^{n_\alpha^0} - q^{-n_\alpha^0}}$$

$$M_1^c(q,q_1)=\prod_{\alpha\in\Delta_1^+} q_1 q^{n_\alpha^c}-q_1^{-1}q^{-n_\alpha^c},$$

then $M_0^c(q)\in\mathbb{Z}[\ q^{\pm1}]$ *and* $M_1^c(q,q_1)\in\mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1}]$.
2) The following relations holds:

$$\frac{\varphi_{\lambda_a^c+\rho}(L_0')}{\varphi_\rho(L_0')}=M_0^c(e^{\frac{h}{2}})\qquad \varphi_{\lambda_a^c+\rho}(L_1')=M_1^c(e^{\frac{h}{2}},e^{\frac{ha}{2}}).$$

3) As a consequence:

$$d(\lambda_a^c)=\frac{M_0^c(e^{\frac{h}{2}})}{M_1^c(e^{\frac{h}{2}},e^{\frac{ha}{2}})}.$$

Lemma 4.1.3 *The module* $\tilde{V}_a^c$ *has a basis* B_a^c *and there exist matrices* $R^{c,d}(x,y,z)\in GL(\mathbb{Z}[\ x^{\pm1},y^{\pm1},z^{\pm1}]\ )$ *such that the* $R-matrix$ *act on* $\tilde{V}_a^c\otimes\tilde{V}_b^d$ *in the basis* $B_a^c\times B_b^d$ *with the matrix* $q^{<\lambda_a^c,\lambda_b^d>}R^{c,d}(q,q^a,q^b)$.

4.2 The multivariable invariants

With the above results, we are ready to introduce the multi-variable invariants.

Theorem 4.2.1 *Consider* L *a link with* k *components which are ordered and colored with elements* $\bar{c}_i\in\mathbb{N}^{r-1}$. *Denote by* $\bar{c}=(\bar{c}_1,...,\bar{c}_k)$. *Then there is a Laurent polynomial in many variables* $M(L,\bar{c})$ *such that:*

1)

$$M(L,\bar{c})\in\begin{cases} M_1^{\bar{c}_1}(q,q_1)^{-1}\mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1}] & \text{if } k=1\\ \mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1},...,{q_k}^{\pm1}] & \text{if } k\geq 2\end{cases}$$

2)For L' *a framing on* L *and* $(\xi_1,...,\xi_k)\in\mathbb{T}_{\bar{c}_1}\times...\times\mathbb{T}_{\bar{c}_k}$, *if we color the* $i'th$ *knot from* L' *with* $\tilde{V}_{\xi_i}^{\bar{c}_i}$ *then:*

$$F'(L')=e^{\sum lk_{i,j}<\lambda_{\xi_i}^{\bar{c}_i},\lambda_{\xi_j}^{\bar{c}_j}+2\rho>\frac{h}{2}}\ M(L,\bar{c})\mid_{q_i=e^{\frac{\xi_i h}{2}}}.$$

Proof: Let us consider $\xi_1,...,\xi_k\in\mathbb{C}$ such that $1,\xi_1,...,\xi_k$ is a linearly independent subset of $\mathbb{C}$ as a $\mathbb{Q}$-vector space.
We'll use this subset and F' in order to define the multivariable invariant $M(q,q_1,...,q_n)$ and after that, we'll show that this does not depend on the

choice of the linearly independent family. Denote by L' the link L colored with the corresponding weights associated to c and $\{\xi_i\}$.
Denote $\phi : \mathbb{Z}[\ q^{\pm 1}, q_1{}^{\pm 1}, ..., q_k^{\pm 1}] \ \to \mathbb{C}[\ [\ h]\ ]$ the application defined by:

$$\phi(q) = e^{\frac{h}{2}}$$

$$\phi(q_i) = e^{\frac{\xi_i h}{2}}.$$

Property: The family $\{q^{t_0} q_1{}^{t_1} ... q_k{}^{t_k} \mid t_i \in \mathbb{Z}, \forall i \in 0...k\}$ is free in $\mathbb{C}[\ [\ h]\ ]$.
From this, it results that ϕ is an injective map.
Let $\check{T}_t$ the $(1,1)$ tangle obtained from L' by cutting the t'th strand.
From the definition of F', we have that: $F'(L') = d(\lambda_{\xi_t}^{\bar{c}_t}) < \check{T}_t >$.
We have that:

$$d(\lambda_{\xi_t}^{c_t}) = \frac{M_0^{c_t}(e^{\frac{h}{2}})}{M_1^{c_t}(e^{\frac{h}{2}}, e^{\frac{\xi_t h}{2}})} \quad (1)$$

Now, we will investigate $< \check{T}_t >$.
In order to do that, it is sufficient to look at $F(< \check{T}_j >)(v_j)$, where v_j is a vector of $V_{\xi_j}^{c_j}$.
For $\forall i \in 1, ..., k$, let $B_{\xi_i}^{c_i}$ the basis that exist from the statment of the *Lemma* 4.1.3.
We can split in "floors" the diagram of L' such that for any floor, each strand is involved in at most one crossing or one curl (up or down).
Using the functoriality and the monoidality of the Reshetikhin-Turaev function, essentially we'll have to look at all local relations and to compute the action of the brading C, the twist θ or the dualities b,d.
Let us fix a crossing corresponding to weights $\lambda_{\xi_i}^{c_i}$ and $\lambda_{\xi_j}^{c_j}$.
Having in mind that we are working the proper basis, the brading, which essentially is given by the action of the R-matrix, will act with the coefficient $q^{<\lambda_{\xi_i}^{c_i}, \lambda_{\xi_j}^{c_j}>}$ and one combination of $\phi(polynomials)$.
Remark: this combination comes from the fact that R is a matrix of polynomials in q, q_i, q_j and when it has to act on $V_{\xi_i}^{c_i} \otimes V_{\xi_j}^{c_j}$, it has to be evaluated with the corresponding colors $R(q, q^{\xi_i}, q^{\xi_j})$.
Conclusion: at each crossing between different strands i and j, the brading acts with the coefficient $q^{<\lambda_{\xi_i}^{c_i}, \lambda_{\xi_j}^{c_j}>}$ and beside this, there are combinations of terms that are in the image of ϕ. Let us look at the set of all crossings between different strands (without the strands framings). For each i, j, consider all the crossings between those strands. At each of those, the brading

will act with the R-matrix or its inverse, depending on the positivity of the crossing. So, at each of them, the brading acts with $q^{\epsilon<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}>}$, where ϵ =sign of the crossing. If we sum up, along all the crossings corresponding to strands i and j, we obtain the coefficient $q^{lki,j<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}>}$ (2).
It remains to look at the crossings coming from the framings of the components.
Let us fix one strand i. From the *Lemma* 4.1.1, we know that the functor F acts locally for one positive framing with the coefficient $q^{<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>}$.
So, overall, we get from the ith strand the coefficient $q^{fr_i<\lambda^{c_i}_{\xi_i},\lambda^{c_i}_{\xi_i}+2\rho>} = q^{lk_{i,i}<\lambda^{c_i}_{\xi_i},\lambda^{c_i}_{\xi_i}+2\rho>}$ (3)
Observation: From the fact that we are working with super Lie algebras of type I, we have $<\lambda^{c_i}_{\xi_i},2\rho>\in \mathbb{Z}+\xi_i\mathbb{Z}$, so $e^{<\lambda^{c_i}_{\xi_i},2\rho>\frac{h}{2}}\in Im(\phi)$.
From relations (2),(3) and from the observation, we obtain that:

$$<\check{T}>\in e^{\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}Im(\phi) \quad (4)$$

Using the facts (1) and (4) and the previous conclusions, we obtain that:

$$F'(L')\in e^{\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}\frac{\phi(M^{c_t}_0(q))}{\phi(M^{c_t}_1(q,q_t))}Im(\phi) \quad (5)$$

Moreover

$$e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}M^{c_t}_1(e^{\frac{h}{2}},\frac{\xi_t h}{2})F'(L')\in Im(\phi)$$

has the form $\phi(M_c)$ where $M_c\in Z[\ q^{\pm1},q_1{}^{\pm1},...,q_k^{\pm1}]$ depends just on the semicolors $c=(c_1,...,c_k)$, but not on the final parts $\xi=(\xi_1,...,\xi_k)$. (6)
Let us denote by

$$M_t(L,c,\xi,fr)=\frac{\phi^{-1}(e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}M^{c_t}_1(e^{\frac{h}{2}},\frac{\xi_t h}{2})F'(L'))}{M^{c_t}_1(q,q_t)}. \quad (7)$$

Here, fr means the framings of L'.
Using the fact that F' is a well defined link invariant from $Theorem$3.2.8, we get that $M_t(L,c,\xi,fr)$ does not depend on t.
From the relation (6), $M_t(L,c,\xi)$ does not depend on the final colors ξ with the condition of being a $\mathbb{Q}$-linearly independent family together with 1.

Because of the term $e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>}$ from the definition above, it follows that $M_t(L,c,\xi,fr)$ does not depend on the framing fr of L'.
So there is a well defined link invariant: $M(L,c):=M_1(L,c,\xi,fr)$.
It is clear that M and F' satisfy the relation from the statement for any family ξ which has linearity independent condition.
But from the fact that $\phi\circ M(L,c)$ and $F'(L')$ depend continuously of ξ, it results that $M(L,c)$ and $F'(L')$ satisfy the desired relation for any $\xi=(\xi_1,...,\xi_k)\in\mathbb{T}_{c_1}\times...\times\mathbb{T}_{c_k}$.

It remains to prove that for $k\geq 2$, $M(L,c)\in\mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1},...,{q_k}^{\pm1}]$.
Having in mind this, we will prove that: $e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}F'(L')\in Im(\phi)$.
Let us denote by $\check{T}_t$ the $(1,1)$-tangle obtained from L' by cutting the t'th strand, $t\in\{1,2\}$. From relation (5), it means that there are polynomials $P_1,P_2\in\mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1},...,{q_k}^{\pm1}]$ such that:

$$e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}F'(L')=d(\lambda^{c_t}_{\xi_t})\phi(P_t),\quad t\in\{1,2\}.$$

Using the expression of d (relation (1)), we obtain the following:

$$e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}F'(L')=\frac{M^{c_1}_0(e^{\frac{h}{2}})}{M^{c_1}_1(e^{\frac{h}{2}},e^{\frac{\xi_1 h}{2}})}\phi(P1)=\frac{M^{c_2}_0(e^{\frac{h}{2}})}{M^{c_2}_1(e^{\frac{h}{2}},e^{\frac{\xi_2 h}{2}})}\phi(P2)$$

$$\Rightarrow M^{c_1}_0(q)M^{c_2}_1(q,q_2)P_1=M^{c_2}_0(q)M^{c_1}_1(q,q_1)P_2.$$

Proposition : $M^{c_1}_1(q,q_1)$ and $M^{c_1}_0(q)M^{c_2}_1(q,q_2)$ are relatively prime in the unique factorization domain $\mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1},...,{q_k}^{\pm1}]$.
From the previous two relations, we obtain that $M^{c_1}_1(q,q_1)$ divides P_1.
We can conclude now that $e^{-\sum_{i,j}<\lambda^{c_i}_{\xi_i},\lambda^{c_j}_{\xi_j}+2\rho>\frac{h}{2}}F'(L')\in Im(\phi)$.
The relation (7) implies that $M(L,c)\in\mathbb{Z}[\ q^{\pm1},{q_1}^{\pm1},...,{q_k}^{\pm1}]$ which finishes the proof.

□

Chapter 5

Examples

This chapter is devoted to examples of computations for some particular knots and links.

5.1 Important facts about $sl(2|1)$

In [7], the authors give explicit formulas for the R-matrix, S', d and some particular representations in the case of super-Lie algebra $sl(2|1)$.

First of all, we will say few words about the general representation theory in this set up. In this case, we have the weight lattice indexed by $\Lambda = \mathbb{N} \times \mathbb{C}$. Then a weight $\alpha = (a_1, a_2) \in \Lambda$ is typical if and only if $a_1 + a_2 + 1 \neq 0$ and $a_2 \neq 0$. Consider $S_1 \subseteq \Lambda$ the set of atypical weights.

Secondly, there are the following formulas for S' and for d:

Proposition 5.1.1 *For $\alpha = (a_1, a_2) \in \Lambda \setminus S_1$ and $\beta = (b_1, b_2) \in \Lambda$ we have:*

$$S'(\alpha, \beta) = q^{-(2a_2+a_1+1)(2b_2+b_1+1)} \frac{\{(a_1+1)(b_1+1)\}}{\{b_1+1\}} \{b_2\}\{b_2+b_1+1\}$$

$$d(\alpha) = \frac{\{a_1+1\}}{\{1\}\{a_2\}\{a_2+a_1+1\}}.$$

For the quantization, $U_h(sl(2|1))$ they explain its R-matrix.
Let as denote $E' := E_1E_2 - q^{-1}E_2E_1$ and $F' := F_2F_1 - qF_1F_2$.
We will used the following definitions:

$$exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!}, \qquad (k)_q = \frac{1-q^k}{1-q}, \qquad (n)_q! = (1)_q \cdot \ldots \cdot (n)_q$$

$$\{1\} = q - q^{-1}, \qquad \{z\} = q^z - q^{-z}$$

Proposition 5.1.2 *The R-matrix has the form* $R = \check{R}K$ *where:*

$$\check{R} = exp_q(\{1\}E_1 \otimes F_1)exp_q(-\{1\}E' \otimes F')exp_q(-\{1\}E_2 \otimes F_2)$$

$$K = q^{-h_1\otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2}.$$

Also there is a very precise description for one family of representations of $U_h(sl(2|1))$.

Proposition 5.1.3 *Let* $a_2 \in \mathbb{C} \setminus \{0, -1\}$ *and the weight* $a = (0, a_2)$*. Then the corresponding* $U_h(sl(2|1))$ *weight module with highest weight* a *has the following form:* $V =< v_1, v_2, v_3, v_4 >$ *such that* $v_1 \leftrightarrow v_{0,0}, v_2 \leftrightarrow w_{1,0}, v_3 \leftrightarrow w_{0,0}, v_4 \leftrightarrow v_{1,0}$ *with the super-grading* $\bar{v_i} = i \bar{+} 1$ *and the elements that generate* $U_h(sl(2|1))$ *act on* V *in this basis as follows:*

$$h_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad h_2 = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_2+1 & 0 & 0 \\ 0 & 0 & a_2+1 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 & q^{-a_2} \cdot \frac{\{a_2\}}{\{1\}} \\ 0 & 0 & \frac{\{a_2+1\}}{\{1\}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q^{a_2} & 0 & 0 & 0 \end{pmatrix}.$$

We will compute how the R-matrix acts on the tensor power of V. We have:

$$E_1E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\{a_2+1\}}{\{1\}} & 0 \end{pmatrix} \quad E_2E_1 = \begin{pmatrix} 0 & q^{-a_2} \cdot \frac{\{a_2\}}{\{1\}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F_2F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad F_1F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q^{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so $E' = \begin{pmatrix} 0 & -q^{-(a_2+1)} \cdot \frac{\{a_2\}}{\{1\}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\{a_2+1\}}{\{1\}} & 0 \end{pmatrix} \quad F' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -q^{(a_2+1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

We observe that all $E_1, F_1, E', F', E_2, F_2$ are nilpotent, namely at the second power they become zero, so we obtain the following expression:

$$\check{R} = (Id + \{1\}E_1 \otimes F_1) \circ (Id - \{1\}E' \otimes F') \circ (Id - \{1\}E_2 \otimes F_2).$$

Denote by A,B,C' the terms that appear in the above formula.

Also, in [7] there is an explicit description action of duality for V:

Proposition 5.1.4

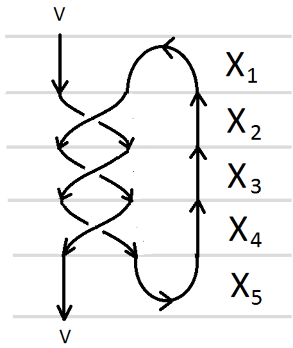
$$d'_V(v_{j,i} \otimes v^*_{j,i}) = (-1)^{\bar{j}} q^{-2a_2-2i}$$

$$d'_V(w_{j,i} \otimes w^*_{j,i}) = (-1)^{\bar{j}} q^{-2a_2-2i-2}.$$

We obtain the following:
$d'(v_1 \otimes v^*_1) = d'(v_{0,0} \otimes v^*_{0,0}) = q^{-2a_2}$
$d'(v_2 \otimes v^*_2) = d'(w_{1,0} \otimes w^*_{1,0}) = -q^{-2a_2-2}$
$d'(v_3 \otimes v^*_3) = d'(w_{0,0} \otimes w^*_{0,0}) = q^{-2a_2-2}$
$d'(v_4 \otimes v^*_4) = d'(v_{1,0} \otimes v^*_{1,0}) = -q^{-2a_2}$.

5.2 The trefoil knot

In this part, we will compute the multivariable invariant for the left trefoil knot.
In order to obtain the multivariable invariant, we will compote the modified invariant F' on the framed trefoil with the blackboard framing. First of all, let us consider the tangle $\check{T}$ obtained from the trefoil knot by cutting its strand.



The first step is to compute the Reshetikhin-Turaev functor evaluated on this tangle. In order to do this, we will use its functoriality and monoidality. Lst us denote by: $f_1 = F(X_1) = id_V \otimes d'_V$, $f_2 = F(X_2) = C_{V,V} \otimes id_{V^*}$, $f_3 = F(X_3) = C_{V,V} \otimes id_{V^*}$, $f_4 = F(X_4) = C_{V,V} \otimes id_{V^*}$, $f_5 = F(X_5) = id_V \otimes b_V$. We will evaluate $F(\check{T})$ on the highest vector v_1.

$$F(\check{T})(v_1) = F(X_1 \circ X_2 \circ X_3 \circ X_4 \circ X_5)(v_1) =$$
$$= F(X_1) \circ F(X_2) \circ F(X_3) \circ F(X_4) \circ F(X_5) = f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5(v_1) =$$
$$= f_1 \circ f_2 \circ f_3 \circ f_4(v_1 \otimes v_1 \otimes v_1^* + v_1 \otimes v_2 \otimes v_2^* + v_1 \otimes v_3 \otimes v_3^* + v_1 \otimes v_4 \otimes v_4^*)$$

.

Now, we have to know which is the action of the brading C on the first two components of the above tensor products.

$$\bullet C(v_1 \otimes v_1) = \tau \circ R(v_1 \otimes v_1) = \tau \circ \check{R}K(v_1 \otimes v_1) =$$
$$= \tau \circ \check{R}(q^{-h_1 \otimes h_2 - h_2 \otimes h_1 - 2h_2 \otimes h_2})(v_1 \otimes v_1) =$$
$$= \tau(q^{-h_1(v_1) \otimes h_2(v_1) - h_2(v_1) \otimes h_1(v_1) - 2h_2(v_1) \otimes h_2(v_1)} v_1 \otimes v_1) =$$
$$= q^{-2a^2} v_1 \otimes v_1$$
$$\bullet C(v_1 \otimes v_2) = \tau(q^{a-2a(a+1)} v_1 \otimes v_2) = q^{-2a^2-a} v_2 \otimes v_1$$
$$\bullet C(v_1 \otimes v_3) = q^{-2a(a+1)} v_3 \otimes v_1$$
$$\bullet C(v_1 \otimes v_4) = q^{-a-2a^2} v_4 \otimes v_1.$$

Observation: we have used in the previous relations that v_1 is a highest weight vector, so the action of R on $v_1 \otimes \cdot$ is essentially the action of K.

Using these formulas in our initial computation, we have:

$$F(\check{T})(v_1) =$$

$$= f_1\circ f_2\circ f_3(q^{-2a^2}v_1\otimes v_1\otimes v_1^*+q^{-a-2a^2}v_2\otimes v_1\otimes v_2^*+q^{-2a(a+1)}v_3\otimes v_1\otimes v_3^*+q^{-a-2a^2}v_4\otimes v_1\otimes v_4^*)$$

For f_3, we need to do again some C-action computations:

$$\bullet C(v_2\otimes v_1) = \tau\check{R}K(v_2\otimes v_1)) = \tau\check{R}(q^{-h_1\otimes h_2-2h_2\otimes h_2}(v_2\otimes v_1)) =$$

$$= \tau q^{a-2a(a+1)}\check{R}((v_2\otimes v_1)) = q^{-2a^2-a}\tau(A\circ B\circ C'((v_2\otimes v_1)))$$

But: $C'(v_2\otimes v_1) = v_2\otimes v_1$, $B\circ C(v_2\otimes v_1) = v_2\otimes v_1 - \{a\}v_1\otimes v_2$,
so, we obtain that: $A(B(C'(v_2\otimes v_1))) = v_2\otimes v_1 - \{a\}v_1\otimes v_2$.

$$\Rightarrow C(v_2\otimes v_1) = q^{-2a^2-a}(v_2\otimes v_1 - \{a\}v_1\otimes v_2).$$

$$\bullet C(v_3\otimes v_1) = \tau\check{R}q^{-2h_2\otimes h_2}(v_3\otimes v_1) = \tau\check{R}q^{-2a(a+1)}(v_3\otimes v_1)$$

$$C'(v_3\otimes v_1) = v_3\otimes v_1 - \{a+1\}q^a v_2\otimes v_4$$

$$B(C'(v_3\otimes v_1)) = v_3\otimes v_1 - q^a\{a+1\}v_2\otimes v_4 + q^{a+1}\{a+1\}v_4\otimes v_2 - q^{-1}\{a\}\{a+1\}v_1\otimes v_3$$

$$A(B(C'(v_3\otimes v_1))) = v_3\otimes v_1 - q^a\{a+1\}v_2\otimes v_4 + q^{a+1}\{a+1\}v_4\otimes v_2 - q^{-1}\{a\}\{a+1\}v_1\otimes v_3$$
$$-q^a\{1\}\{a+1\}v_4\otimes v_2$$

$$\Rightarrow C(v_3\otimes v_1) = q^{-2a(a+1)}v_1\otimes v_3 + q^{-2a^2-a}\{a+1\}v_4\otimes v_2 - q^{(a+1)(1-2a)}\{a+1\}v_2\otimes v_4$$
$$-q^{-2a(a+1)-1}\{a\}\{a+1\}v_3\otimes v_1 + q^{-2a^2-a}\{a+1\}\{1\}v_2\otimes v_4.$$

$$\bullet C(v_4\otimes v_1) = \tau\check{R}K(v_4\otimes v_1) = \tau\check{R}q^{-a-2a^2}(v_4\otimes v_1)$$

But: $C'(v_4\otimes v_1) = v_4\otimes v_1 - \{a\}v_1\otimes v_4$, $B(C'(v_4\otimes v_1)) = v_4\otimes v_1 - \{a\}v_1\otimes v_4$
and $A(B(C'(v_4\otimes v_1))) = v_4\otimes v_1 - \{a\}v_1\otimes v_4$.

$$\Rightarrow C(v_4\otimes v_1) = q^{-2a^2-a}v_1\otimes v_4 - q^{-2a^2-a}v_4\otimes v_1$$

We have that:

$$F(\check{T})(v_1) = f_1\circ f_2(q^{-4a^2}v_1\otimes v_1\otimes v_1^* + q^{-4a^2-2a}v_1\otimes v_2\otimes v_2^* -$$

$$-q^{-4a^2-2a}\{a\}v_2\otimes v_1\otimes v_2^* + q^{-4a(a+1)}v_1\otimes v_3\otimes v_3^* +$$

$$+q^{-4a^2-3a}\{a+1\}v_4\otimes v_2\otimes v_3^* - q^{(a+1)(1-4a)}\{a+1\}v_2\otimes v_4\otimes v_3^* -$$

$$-q^{-4a(a+1)-1}\{a\}\{a+1\}v_3\otimes v_1\otimes v_3^*+q^{-4a^2-3a}\{a+1\}\{1\}v_2\otimes v_4\otimes v_3^*+$$

$$+q^{-4a^2-2a}v_1\otimes v_4\otimes v_4^*-q^{-4a^2-2a}\{a\}v_4\otimes v_1\otimes v_4^*)$$

In order to compute the value of f_2 on the previous sum, we have to know C acts on $v_2\otimes v_4$ and $v_4\otimes v_2$.

$\bullet C(v_4\otimes v_2)=\tau\check{R}K(v_4\otimes v_2)=\tau\check{R}q^{-(a+1)+a-2a(a+1)}(v_4\otimes v_2)=\tau\check{R}q^{-2a(a+1)-1}(v_4\otimes v_2)$

But: $C'(v_4\otimes v_2)=v_4\otimes v_2-q^{-a}\{a\}v_1\otimes v_3$
$B(C'(v_4\otimes v_2))=v_4\otimes v_2-q^{-a}\{a\}v_1\otimes v_3$
$A(B(C'(v_4\otimes v_2)))=v_4\otimes v_2-q^{-a}\{a\}v_1\otimes v_3$

$$\Rightarrow C(v_4\otimes v_2)=q^{-2a(a+1)-1}(\tau(v_2\otimes v_4)-q^{-a}\{a\}\tau(v_1\otimes v_3))=$$

$$-q^{-2a(a+1)-1}v_2\otimes v_4-q^{-2a^2-3a-1}\{a\}v_3\otimes v_1$$

$$\bullet C(v_2\otimes v_4)=\tau\check{R}K(v_2\otimes v_4)=\tau\check{R}q^{-2a(a+1)-1}(v_2\otimes v_4)$$

Also: $C'(v_2\otimes v_4)=v_2\otimes v_4$
$B(C'(v_2\otimes v_4))=v_2\otimes v_4+q^{-(a+1)}\{a\}v_1\otimes v_3$
$A(B(C'(v_2\otimes v_4)))=v_2\otimes v_4+q^{-(a+1)}\{a\}v_1\otimes v_3+\{1\}v_4\otimes v_2$

$$\Rightarrow C(v_2\otimes v_4)=q^{-2a(a+1)-1}(\tau(v_2\otimes v_4)+q^{-(a+1)}\{a\}\tau(v_1\otimes v_3)+\{1\}\tau(v_4\otimes v_2))=$$

$$=-q^{-2a(a+1)-1}v_4\otimes v_2+q^{-2a^2-3a-2}\{a\}v_3\otimes v_1-q^{-2a(a+1)-1}\{1\}v_2\otimes v_4$$

We obtain that:

$$F(\check{T})(v_1)=f_1(...+q^{-6a^2}v_1\otimes v_1\otimes v_1^*)-q^{-6a^2-3a}\{a\}v_1\otimes v_2\otimes v_2^*-$$

$$-q^{-6a(a+1)-1}\{a\}\{a+1\}v_1\otimes v_3\otimes v_3^*-q^{-6a^2-3a}\{a\}v_1\otimes v_4\otimes v_4^*).$$

To apply the function f_1, we'll use the formulas for d' from the end of the introductory part of this chapter. This is the reason for which we did not write all the terms in the previous computation, the non zero elements obtained from f_1, are those which have the same indexes on the last two positions in the tensor product.

$$\Rightarrow F(\check{T})(v_1)=(q^{-6a^2-2a}+q^{-6a^2-3a}(q^a-q^{-a})q^{-2a-2}-$$

$$-q^{-6a(a+1)-1}(q^a-q^{-a})(q^{a+1}-q^{-(a+1)})q^{-2a-2}+q^{-6a^2-3a}(q^a-q^{-a})q^{-2a})\cdot v_1.$$

It means that the bracket has the following form: (now, we will return to the correct notation $a\to a_2$)

$$<\check{T}>=q^{-6a_2^2-2a_2}(1+q^{-2a_2-2}-2q^{-4a_2-2}+q^{-6a_2-4}+q^{-6a_2-2}-q^{-8a_2-4}+q^{-2a_2}-q^{-4a_2}).$$

From $Proposition 5.1.1$, the modified dimension has the following form:

$$d((0,a_2)) = \frac{\{1\}}{\{1\}\{a_2\}\{a_2+1\}} = \frac{1}{(q^{a_2}-q^{-a_2})(q^{a_2+1}-q^{-(a_2+1)})}.$$

We obtain the expression of the modified invariant:

$$F'(T) = d((0,a_2)) < \check{T} >=$$

$$\frac{q^{-6a_2^2-2a_2}(1+q^{-2a_2-2}-2q^{-4a_2-2}+q^{-6a_2-4}+q^{-6a_2-2}-q^{-8a_2-4}+q^{-2a_2}-q^{-4a_2})}{(q^{a_2}-q^{-a_2})(q^{a_2+1}-q^{-(a_2+1)})}$$

Now, using the relation between F' and M, we are able to compute the multivariable invariant.
Having in mind that we chose the trefoil with the blackboard framing, it means that $lk(T,T)=3$. We have the following relation:

$$F'(T) = q^{-3(2a_2^2+2a_2)} M(T)(q, q^{a_2}).$$

It means that:

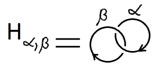
$$M(T)(q,q^{a_2}) = \frac{q^{4a_2}+q^{2a_2-2}-2q^{-2}+q^{-2a_2-4}+q^{-2a_2-2}-q^{-4a_2-4}+q^{2a_2}-1}{(q^{a_2}-q^{-a_2})(q^{a_2+1}-q^{-(a_2+1)})}.$$

We obtain the formula for the multivariable invariant evaluated on the left-trefoil knot:

$$M(T)(q,q_1) = \frac{{q_1}^4+{q_1}^2q^{-2}-2q^{-2}+{q_1}^{-2}q^{-4}+{q_1}^{-2}q^{-2}-{q_1}^{-4}q^{-4}+{q_1}^2-1}{(q_1-{q_1}^{-1})(qq_1-(qq_1)^{-1})}.$$

5.3 The Hopf and multi-Hopf links

We will compute the multivariable invariant for the Hopf link with framing zero in each component and colored with two weights $\alpha = (a_1,a_2), \beta = (b_1,b_2) \in \Lambda \setminus S_1$. Let us make the following notation:



.

Applying the formulas from the beginning of this chapter we have:

$$F'(H_{\alpha,\beta}) = S'(\alpha,\beta)d(\beta) =$$

$$= \frac{q^{-(2a_2+a_1+1)(2b_2+b_1+1)}\frac{\{(a_1+1)(b_1+1)\}}{\{b_1+1\}}\{b_2\}\{b_2+b_1+1\}\{b_1+1\}}{\{1\}\{b_2\}\{b_2+b_1+1\}}$$

$$\Rightarrow F'(H_{\alpha,\beta}) = \frac{q^{-(2a_2+a_1+1)(2b_2+b_1+1)}\{(a_1+1)(b_1+1)\}}{\{1\}}.$$

We have to color with colors of type $\alpha_0 = (0, a_2)$ and $\beta_0 = (0, b_2)$ in order to obtain the multivariable invariants for the uncolored framed Hopf link:

$$F'(H_{\alpha_0,\beta_0}) = q^{-(2a_2+1)(2b_2+1)}$$

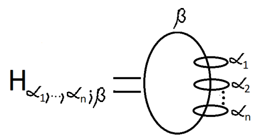
From $Theorem 2$ from [7] we have:

$$F'(H_{\alpha_0,\beta_0}) = q^{-(lk_{1,2}(2a_2b_2+a_2+b_2)+lk_{2,1}(2b_2a_2+b_2+a_2))} \cdot M(H)(q, q^{a_2}, q^{b_2})$$

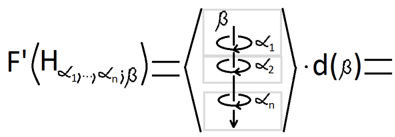
Using the previous two relations, we get:

$$M(H)(q, q^{a_2}, q^{b2}) = q^{-1}, \text{ so } M(H)(q, q_1, q_2) = q^{-1}.$$

Let us consider the multi-Hopf link:



In order to compute the invariant we have:



$$= (\prod_{i=1}^{n} S'(\alpha_i,\beta))d(\beta) =$$

$$= \prod_{i=1}^{n}(q^{-(2a_2^i+a_1^i+1)(2b_2+b_1+1)}\frac{\{(a_1^i+1)(b_1+1)\}}{\{b_1+1\}}\{b_2\}\{b_2+b_1+1\})\cdot\frac{\{b_1+1\}}{\{1\}\{b_2\}\{b_2+b_1+1\}}.$$

As a conclusion, the modified invariant evaluated on the colored multi-Hopf link has the following value:

$$F'(H_{\alpha_1,...,\alpha_n;\beta}) =$$

$$= q^{-(\sum_{i=1}^{n}(2a_2^i+a_1^i+1))(2b_2+b_1+1)} \cdot \prod_{i=1}^{n}\{(a_1^i+1)(b_1+1)\} \cdot (\frac{\{b_2\}\{b_2+b_1+1\}}{\{b_1+1\}})^{n-1} \cdot \frac{1}{\{1\}}.$$

For the multivariable invariants, we compute F' on the multi-Hopf link with colors $\alpha_i^0 = (0, a_2^i), i \in 1, \bar{...}, n$ and $\beta^0 = (0, b_2)$:

$$F'(H_{\alpha_1^0,...,\alpha_n^0;\beta^0}) = q^{-(\sum_{i=1}^{n}(2a_2^i+1))(2b_2+1)}(\{b_2\}\{b_2+1\})^{n-1}.$$

But, on the other hand we have:

$$F'(H_{\alpha_1^0,...,\alpha_n^0;\beta^0}) = q^{(-2\sum_{i=1}^{n} lk_{i,n+1}(2a_2^i b_2 + a_2^i + b_2))} \cdot M(H_{n,1})(q, q^{a_2^1}, ..., q^{a_2^n}, q^{b_2}).$$

From the last two relations, we get:

$$M(H_{n,1})(q, q^{a_2^1}, ..., q^{a_2^n}, q^{b_2}) = q^{-n}((q^{b_2} - q^{-b_2})(q^{b_2+1} - q^{-(b_2+1)}))^{n-1}.$$

After a change of variables, we obtain the expression of the multivariable invariants on the framed zero multi-Hopf link:

$$M(H_{n,1})(q, q_1, ..., q_n, q_{n+1}) = q^{-n} \cdot (q_{n+1} - {q_{n+1}}^{-1})^{n-1} \cdot (qq_{n+1} - q^{-1}q_{n+1}^{-1})^{n-1}.$$

Appendix

A1. Fundamental facts on Super Lie algebras

Let $g = g_0 \oplus g_1$ a Super Lie algebra

Definition *1) A sub-algebra $g' \subseteq g$ is called solvable if its derived series stabilizes at zero: $g' \supseteq [\ g', g']\ \supseteq ... \supseteq 0$.*
2) A sub-algebra $h \subseteq g_0$ is called nilpotent if its lower central series stabilizes at zero: $h \supseteq [\ h, h]\ \supseteq [\ h, [\ h, h]\ ]\ \supseteq ... \supseteq 0$.

Definition *1) We call $b \subseteq g$ a Borel sub-algebra if it is maximal solvable and $\exists\ b_0 \subseteq g_0$ which is maximal solvable such that $b_0 \subseteq b$.*
2) A sub-algebra $\mathfrak{H} \subseteq g_0$ which is self-normalizing ($[\ x, \mathfrak{H}]\ \subseteq \mathfrak{H} \Rightarrow x \in \mathfrak{H}$) and nilpotent is called a Cartan sub-algebra.

Definition *1)A weight is an element of $\mathfrak{H}^*$.*
2)For a weight $\alpha \in \mathfrak{H}^*$ the weight space is:

$$G_\alpha = \{x \in g \mid [\ h, x]\ = \alpha(h)x, \forall h \in \mathfrak{H}\}$$

.
3)A root is a weight α such that $G_\alpha \neq 0$.
4)A root α is called positive if $G_\alpha \cap n_+ \neq 0$ or negative if $G_\alpha \cap n_- \neq 0$.
5)A root α is called even if $G_\alpha \cap g_0 \neq 0$ or odd if $G_\alpha \cap g_1 \neq 0$.

Proposition *For g a super Lie algebra of type I, for any root $\alpha \in \mathfrak{H}^*$, the weight space G_α has dimension one.*

Definition *The Weyl group $W \subseteq GL(\mathfrak{H}*)$ is defined as the group generated by reflections on hyperplanes orthogonal on roots.*

Proposition *The Weyl group is a finite group.*

Directions for further study

Our aim in the future is to develop a categorification and a geometric description of Heegaard-Floer type for the Geer-Patureau invariants described in this thesis. This section is devoted to a brief description of our plan.

Until their appearance in 2010, the theory of knots and links invariants contained two important directions: the classical one, represented by the Alexander or multivariable Alexander like polynomials and the quantum one, which studied invariants constructed from quantum groups. The classical Alexander invariants, were categorified by the Heegaard-Floer homology, using geometric ideas. Also, the Jones polynomial was categorified by Khovanov homology. The Geer-Patureau multivariable invariants, contain the family of multivariable Alexander polynomials and their construction is based on quantum groups coming from super Lie algebras of type I. Their categorification, which we want to obtain, is something between the classical case and the quantum group case. Having in mind the classical situation, we think at a Floer type construction in order to build the categorification. The first step will be to find a homological interpretation for the Geer-Patureau multivariable invariants. Secondly, using it, we will obtain a Heegaard-Floer type homology which will gives the desired categorification.

The geometric side of our plan is related with a special case of Floer Theory, namely Lagrangian Floer Homology. The initial Floer's idea was to extend the classical Morse-Smale Complex in an infinite dimensional setting. In particular, for a symplectic manifold and two Lagrangian submanifolds, the Floer theory gives a certain 'complex' for which the square of the differential is not always zero. Its construction relies on the intersection points between the two Lagrangian submanifolds and the differential is defined in a geometrical way using J-holomorphic strips. Ozsvath and Szabo applied this construction for a specific symplectic manifold that arise from a Heegaard splitting of a 3-manifold. In this particular case, the Lagrangian Floer complex is well defined and its homology is called the Heegaard Floer homology.

A deeper comprehension of the link invariants appeared in 2000 when Khovanov [13] introduced the categorification in the study of knots and links. The main idea in this type of construction is to obtain a homology theory which has as Euler characteristic the initial invariants, the essential point being the fact that the resulting homology is a more powerful invariant than the ones we started with. His original work categorified the Jones polynomial and it was developed in various ways for some other types of knots or links polynomials. Nowadays, this area of research become a very active one, with remarkables results as Khovanov-Rozansky homology which categorifies the HOMFLY polynomial or Heegaard-Floer homology which categorifies the Alexander polynomial and its multivariable variant for links. Also, a recent result asserts that embedded contact homology is another categorification for the multivariable Alexander polynomial for links. The Geer- Patureau invariants are a missing piece from the above picture, whence the interest for the direction we want to pursue.

The above mentioned Heegaard-Floer Homology is an important theory among all the known type of categorifications. It was introduced by Ozsvath and Szabo in 2004. Their main idea was to use a Floer type construction which comes from symplectic geometry starting with a Heegaard decomposition of a three manifold. The resulting homology groups are powerful invariants, which can detect several knot properties like the genus and fiberedness, can bound the unknotting number and the slice genus and they are related with other knot homology theories like the ones of Khovanov and Rozansky. Moreover, the Heegaard-Floer homology detects the unknot, it is algorithmic computable and beside its original definition which involves J-holomorphic disks, it has a purely combinatorial description. The project that we propose, which aims to find a categorification for the Geer-Patureau multivariable polynomial link invariants is thought to have two main parts. In the first one we are interested to find a topological/geometrical interpretation for this family of multivariable invariants. The original definition of Geer and Patureau is based on purely algebraic techniques coming from the representation theory of super Lie algebras of type I and quantum groups. Therefore, a topological description of the multivariable link invariants would be an interesting goal in its own right. Moreover, our idea is to use this topological interpretation as a starting point for the second part of our construction. More specifically we want to use the topological description in order to define a homology theory which has as Euler characteristic the Geer-Patureau multivariable polynomial link invariants.

In the first part we are looking for a description of the Geer-Patureau invariants using intersection theory of submanifolds. We will search for a symplectic manifold such that our invariants will be obtained by a formula in terms of graded intersections of Lagrangian submanifolds in the ambient symplectic manifold. In some classical cases, a similar idea was used in order to define a categorification for knot invariants. Apart the Heegaard-Floer homology, where this idea was used for the (multivariable) Alexander polynomial cf. [16], [17], [18], a topological intersection type description for the Jones polynomial was obtained by Bigelow [B2] and Lawrence [L2]. Also, very recently Ito [10] obtained an homological description of the colored Alexander polynomial.

Very briefly, the known geometric description of the invariants goes as follows. In the HeegaardFloer case, the authors consider a Heegaard diagram for a three manifold and from it they construct in the symmetric product of the Heegaard surface the lagrangian intersection of multidimensional tori, which are obtained from the Heegaard cut systems. The Bigelows approach to the Jones polynomial uses a certain covering of the configurations space for the punctured disk and two specific sub-manifolds in this covering. Starting from a link viewed as a braid closure, he is looking at the automorphism of the covering induced by this braid. After that, the Jones polynomial is recovered from the topological intersection constructed from the two submanifolds above and the induced automorphism.

In the second part, we plan to use the geometrical description of the invariants as an initial tool towards a Floer type model. Using the graded intersection points, we will define a complex, with the boundary constructed in a geometrical manner. For example, in the definition of the Heegaard-Floer homology, one intersects curves on surfaces, one takes the complex generated by the intersection points and the boundary is introduced geometrically, from one point to another if they bound a bigon. After that, we will consider the homology of the constructed complex which will categorify the Geer-Patureau multivariable link invariants.

We hope that our future work that we propose will have an important impact in our area of interest. Last but not least our project has two sides. First of all, the input is of algebraic nature and it is coming from quantum groups. Secondly, we will use a geometric approach to the subject. This might lead to new interactions between the two domains: Representation Theory and Geometry.

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